IMPROVING THE RESOLUTION PERFORMANCE OF EIGENSTRUCTURE-BASED DIRECTION-FINDING ALGORITHMS*

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ABSTRACT

Recently there has been much interest in algorithms which form a direction-finding spectrum based on the eigenstructure of the sensor covariance matrix. These algorithms are attractive because of their ability to achieve Cramer-Rao direction-finding accuracy bounds for closely spaced emitters, provided the available signal-to-noise (SNR) is high enough to resolve two distinct peaks in the estimated spectrum. In this paper we present several methods for reducing the SNR required for resolution. The first method involves examining the roots of the spectrum polynomial. This technique is applicable when a uniformly spaced sensor array is in use. The second method uses the properties of the so-called signal-space eigenvectors to define a rational (pole-zero) spectrum function with improved resolution capabilities.

INTRODUCTION

Consider an array of M sensors sampling a wavefield arising from L far-field emitters, where L < M. At time t_j, emitter i has associated amplitude $f_i(t_j)$ and direction $\underline{\theta_i}$ with respect to the origin. The received sensor data vector, $\underline{x}(t_j)$, is given by:

$$\underline{x}(t_j) = V\, \underline{f}(t_j) + \underline{W}(t_j)$$

where

$$V = \left(\underline{V}(\underline{\theta}_1) \,.\,.\,.\, \underline{V}(\underline{\theta}_L)\right)$$

$$\underline{f}(t_j) = \left(f_1(t_j) \,.\,.\,.\, f_L(t_j)\right)^T$$

$$\underline{W}(t_j) = \left(W_1(t_j) \,.\,.\,.\, W_M(t_j)\right)^T$$

"T" denotes the transpose operation, $\underline{W}(t_j)$ is the sensor noise vector, and $\underline{V}(\underline{\theta})$ is the steering vector associated with angle $\underline{\theta}$.

Let us assume that the sensor noise is uncorrelated with the emitters, so the sensor covariance matrix is given by:

$$R = E\{\underline{x}\, \underline{x}^H\} = VPV^H + \sigma^2 Q$$

where "H" denotes the conjugate transpose operation, and P is the signal-in-space covariance matrix. For the remainder of this paper, we will assume Q = I.

The basic problem of interest is, given N snapshots of the data how can we estimate the number of emitters, L, the steering vectors V, and their associated bearing angles? In the next section we will present the principles of eigenvector-based direction-finding methods, using Schmidt's MUSIC algorithm [1] as an example.

EIGENVECTOR-BASED DIRECTION-FINDING METHODS [1-3]

Let us assume we have the eigen decomposition of the sensor covariance matrix:

$$R = E_S \Lambda_S E_S^H + E_N \Lambda_N E_N^H$$

where the columns of E_S are the so-called signal eigenvectors; the range of E_S is sometimes referred to as the signal space, the columns of E_N are the so-called noise eigenvectors; the range of E_N is sometimes referred to as the noise space. Eigenvector-based, direction-finding methods make use of the following properties of this decomposition:

$$\Lambda_N = \sigma^2 I_{M-L \times M-L} \quad \text{and}$$

$$\Lambda_S = \text{diag}\,(\lambda_1, \lambda_2, \ldots \lambda_L)$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_L > \sigma^2$$

by convention, and

$$E_N^H\, \underline{V}(\underline{\theta}_i) = \underline{0} \qquad \text{for } i = 1, \ldots L$$

The direction finding process consists of the following: Find the eigenstructure of

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$$\hat{R} = \frac{1}{N} \sum_{j=1}^{N} \underline{x}(t_j)\, \underline{x}^H(t_j) \quad ,$$

the sample covariance matrix. It can be shown that the eigenvectors of $\hat{R}$ are maximum likelihood estimates of the eigenvectors of R [4]. Estimate L using the eigenvalues of $\hat{R}$ in a hypothesis testing procedure [1,4] which, in effect, looks for a cluster of the smallest eigenvalues. Finally, identify candidate direction vectors as the ones which are "farthest" from the noise eigenvectors. The spectrum defined by the methods acts like an inverse distance measure. For MUSIC, the spectrum is given by:

$$S_{MUSIC}(\underline{\theta}) = \frac{1}{\|\hat{\underline{E}}_N^H \underline{V}(\underline{\theta})\|^2} = \frac{1}{\underline{V}^H(\underline{\theta})\hat{\underline{E}}_N \hat{\underline{E}}_N^H \underline{V}(\underline{\theta})}$$

A number of researchers have noted the ability of eigenvector-based, direction-finding methods to achieve the Cramer-Rao accuracy bound for closely-spaced emitters [1, 3], provided the available SNR is high enough for resolution. In the remainder of this paper, we will focus our attention on improving the resolution performance of these methods.

IMPROVED RESOLUTION I - POLYNOMIAL ROOTING

Suppose, for the moment, we restrict our attention to uniform linear arrays with interelement spacing d, so that the m^{th} element of $\underline{V}(\theta)$ may be written as:

$$V_m(\theta) = e^{j2\pi m(d/\lambda)\sin\theta} \quad ; \quad m = 1 \ldots M$$

Let us restrict our attention to "all-pole" direction-finding spectra of the form:

$$S(\theta) = \frac{1}{\underline{V}^H(\theta)\, A\underline{V}(\theta)}$$

where A is Hermitian. For MUSIC, $A = \hat{E}_N \hat{E}_N^H$, for MLM [5], $A = \hat{R}^{-1}$, etc. The denominator may be written as

$$S^{-1}(\theta) = \sum_{m=1}^{M} \sum_{n=1}^{M} e^{-j2\pi m(d/\lambda)\sin\theta} A_{mn} e^{j2\pi n(d/\lambda)\sin\theta}$$

$$= \sum_{\ell=-M+1}^{M-1} a_\ell e^{-j2\pi\ell(d/\lambda)\sin\theta}$$

where a_ℓ is the sum of entries of A along the ℓ^{th} diagonal, i.e.,

$$a_\ell \triangleq \sum_{m-n=\ell} A_{mn}$$

If we define the polynomial:

$$D(z) = \sum_{\ell=-M+1}^{M+1} a_\ell z^{-\ell}$$

then, evaluating the spectrum S(θ) is equivalent to evaluating the polynomial D(z) on the unit circle. Can we use the roots of D(z) for angle-of-arrival estimation rather than searching for peaks in S(θ)? Clearly, peaks in S(θ) are due to roots of D(z) lying close to the unit circle, i.e., a pole of D(z) at

$$z = z_1 = |z_1| e^{j \cdot \arg(z_1)}$$

will result in a peak in S(θ) at

$$\sin(\theta) = \left(\frac{\lambda}{2\pi d}\right) \arg(z_1)$$

Now suppose we have two closely-spaced emitters and we are trying to estimate their angles, but the SNR is low enough so that there is only one peak in the spectrum. Can we obtain reasonable angles estimates for the two emitters using the roots of D(z)? Figure 1, which shows a MUSIC spectrum and its associated poles, suggests that this is a pratical strategy. In this figure, there are three equal-powered signals, two of which are closely spaced. The vertical lines indicate the true locations of the emitters. There is only one peak associated with the closely spaced emitters, yet the roots show their proper locations. Monte Carlo simulations demonstrate that for a given method, the "ROOT" version has better resolution performance than the traditional "SPECTRAL" version.

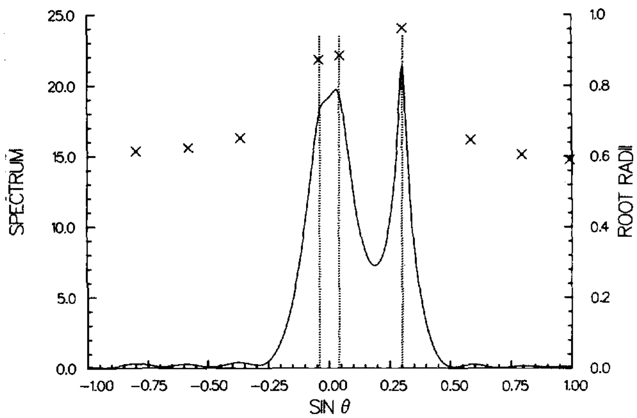

Fig. 1. MUSIC spectrum and associated polynomial roots for equal-powered emitters.

IMPROVED RESOLUTION II - A POLE-ZERO METHOD

Although polynomial-based direction-finding methods offer better resolution performance than the corresponding "spectral" methods, their use is restricted to uniform linear arrays. If we have an arbitrary array, we would like to increase the resolution performance of a given direction-finding method. One approach is to construct a numerator which has a "desirable" relationship to a given denominator. Expressed in the "polynomial" nomenclature, we would like a rational function whose zeroes interlace the poles, yielding a spectrum with peaks in the same location as the poles. Using the properties of the "signal-space" eigenvectors of the sensor covariance matrix, we can construct such a function.

Suppose we have two closely-spaced, equal-powered emitters, and we perform an eigenvector analysis of the sample covariance matrix. In this case, the eigenvector associated with the largest eigenvalue is proportional to the sum of the two steering vectors. If we use "ROOT" MUSIC under the false assumption of one emitter present, we find that there is a single root located near the centroid of the two emitters. When the algorithm is used under the correct assumption of two emitters present, there are two roots, each at the proper signal location.

When there are multiple emitters, and the assumed signal space dimension is smaller than the number of emitters, the roots of the polynomial may correspond to either single, true emitters, or clusters of closely-spaced emitters.

With the above phenomena in mind, we propose the following scheme: Let $\lambda_1 \geq \lambda_2 \ . \ . \ . \geq \lambda_M$ be the eigenvalues of the sample covariance matrix, and let $\underline{e}_i$ be the eigenvector associated with eigenvalue λ_i. Now define:

$$E_{N_L} = [\underline{e}_{L+1} \ \underline{e}_{L+2} \cdots \underline{e}_M] = E_N \quad \text{(the standard noise-space matrix)}$$

$$E_{N_{L-1}} = [\underline{e}_L \ \underline{e}_{L+1} \cdots \underline{e}_M]$$

$$\vdots$$

$$E_{N_1} = [\underline{e}_2 \cdots \underline{e}_M]$$

Define the following direction-finding spectrum:

$$S(\theta) = \left(\frac{\underline{V}^H E_{N_{L-1}} E^H_{N_{L-1}} \underline{V}}{\underline{V}^H E_{N_L} E^H_{N_L} \underline{V}}\right) \left(\frac{\underline{V}^H E_{N_{L-2}} E^H_{N_{L-2}} \underline{V}}{\underline{V}^H E_{N_L} E^H_{N_L} \underline{V}}\right) \cdots \left(\frac{\underline{V}^H E_{N_1} E^H_{N_1} \underline{V}}{\underline{V}^H E_{N_L} E^H_{N_L} \underline{V}}\right) \left(\frac{1}{\underline{V}^H E_{N_L} E^H_{N_L} \underline{V}}\right)^r$$

For the i^{th} factor, L-i zeroes correspond to either true signal zeroes or clusters of signals. The "cluster" zeroes are desirable - they will sharpen the spectrum in the desired fashion, since they lie in between the signal poles. The "signal" zeroes could be a source of potential trouble. If these are closer to the unit circle than their corresponding poles, then the associated peak may be significantly shifted from the proper location. The last factor is included to reduce the chance for these side effects. Clearly, r must not be too large, or the new spectrum will have essentially the same resolution capabilities as the spectral MUSIC algorithm. The simultaneous use of the first L-1 factors is proposed to help improve the stability of the peak locations, although the utility of his strategy versus the potential computation drawbacks needs to be investigated.

Once again, let us turn out attention to Figure 1, which shows a MUSIC spectrum when there are three equal-powered emitters, two being closely spaced. As previously noted, there is only one peak for the two closely-spaced emitters. Figure 2 shows the improved spectrum when the proposed pole-zero/ eigenspace method is used on the same data, for r=1.

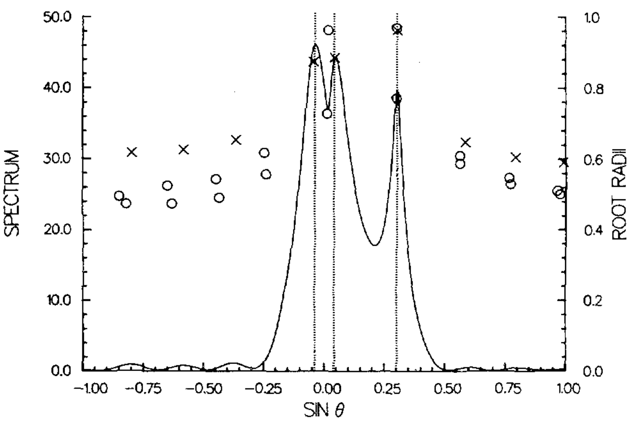

Fig. 2. Proposed method spectrum and associated poles and zeroes (NOTE: all poles are "triple" poles).

SIMULATION RESULTS

Figures 3-6 show Monte Carlo simulation results for MUSIC, "ROOT" MUSIC and the proposed method, for r = 1. The simulations were run using a 10-element, uniformly-spaced, linear array, with interelement spacing of one-half wavelength, and

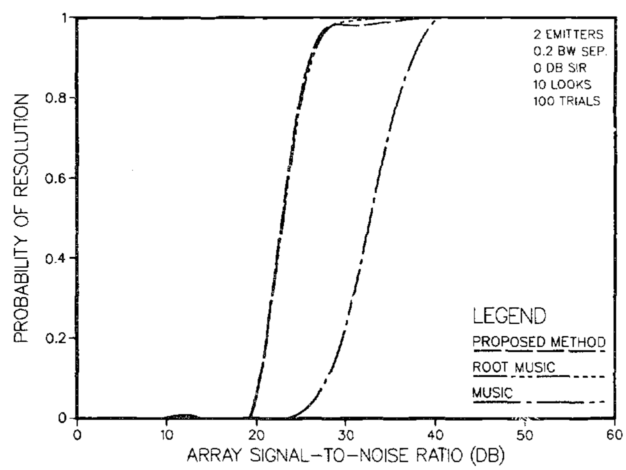

Fig. 3. Resolution performance for two equal-powered emitters.

the sample covariance matrix was formed using 10 looks of the data. Figures 3 and 4 show resolution and direction-finding results for two equal-powered emitters spaced 0.2 beamwidths (BW) apart. Both "ROOT" MUSIC and the proposed method resolve the emitters with 10 dB lower SNR than MUSIC. All the methods yield near optimal angle estimates. Figures 5 and 6 show resolution and

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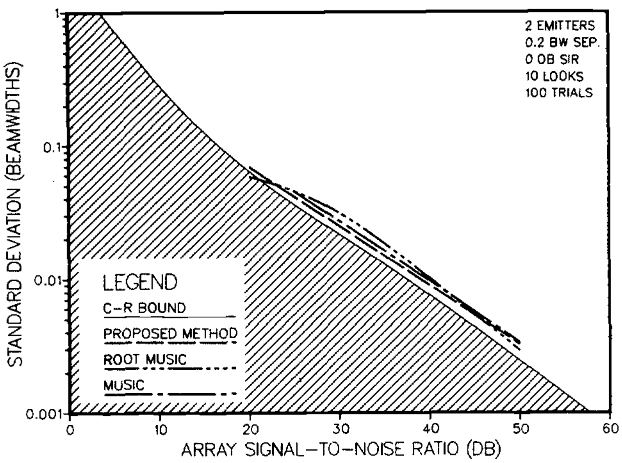

Fig. 4. Angle estimate accuracy for two equal-powered emitters.

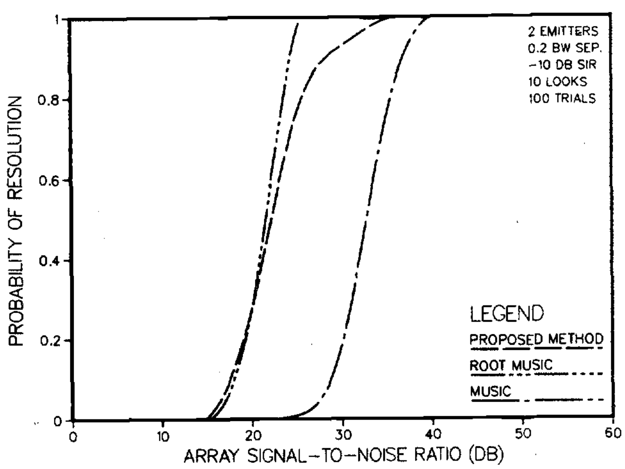

Fig. 5. Resolution performance for two unequal-powered emitters.

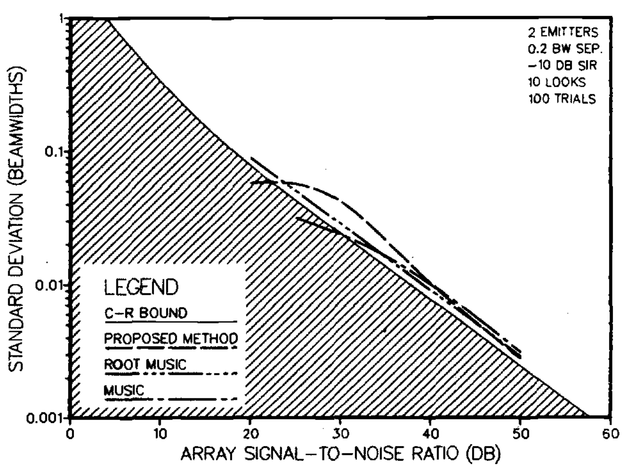

Fig. 6. Angle estimate accuracy for the weaker of two unequal-powered emitters.

direction-finding results for two emitters spaced 0.2 BW apart. In this case, one emitter is 10 dB less powerful than the other. The direction-finding statistics are shown for the weaker emitter. Once again, "ROOT" MUSIC resolves the emitters with 10 dB lower SNR than MUSIC, with both methods yielding near-optimal angle estimates. The proposed method has suffered some performance degradation. The reason for this performance loss is that the largest eigenvector is now aligned with the steering vector associated with the more powerful emitter, resulting in a degradation of our desired pole-zero interlacing property, as shown in Figure 7.

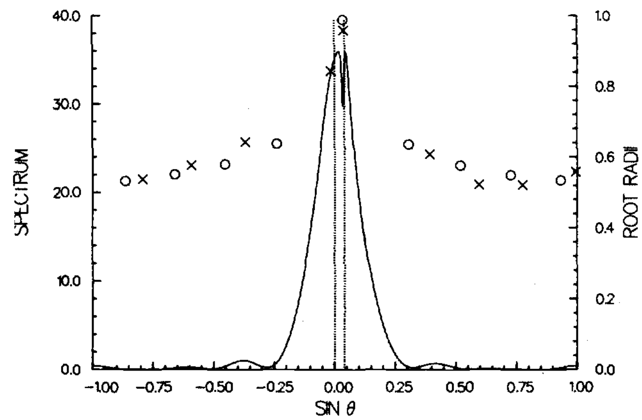

Fig. 7. Proposed method spectrum and associated poles and zeroes for two unequal-powered emitters.

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