MUSIC's and Cramér-Rao Bound in Fourth-Order Cumulant Domain

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Abstract

A unifying asymptotic performance analysis of a class of MUSIC algorithms for direction-of-arrival (DOA) estimation in fourth-order cumulant domain (FOCD-MUSIC) is presented in this paper. A simple and explicit formula for the asymptotic variances of DOA estimation by FOCD-MUSIC's is given. The Cramér-Rao bound for DOA estimation in fourth-order cumulant domain (FOCD-CRB) is also derived. The performances of three typical FOCD-MUSIC's and the conventional covariance-based MUSIC are compared. It is shown that the FOCD-MUSIC's are inefficient and they are not superior to the conventional MUSIC algorithm in any case. Nevertheless, the FOCD-MUSIC's outperform the conventional MUSIC with reduced variances and improved robustness when the spatial sources are closely spaced and the signal-to-noise ratios (SNR's) are relatively low. Simulations are included to validate the analytical results.

1. Introduction

In recent years, DOA estimation algorithms based on higher-order cumulants have drawn a lot of attention [3-9] due to their ability to improve the robustness of the covariance-based techniques. Pan and Nikias [3] were the first to report on the fourth-order cumulant-based MUSIC [2]. They used the diagonal slice to form a spatial cumulant matrix. Porat and Friedlander [4] used all spatial fourth-order cumulants to develop a MUSIC-like algorithm. Moulines and Cardoso [5] proposed the contracted quadricovariance by smoothing part of the spatial fourth-order cumulants. Chen and Lin [6] presented a class of fourth-order cumulant matrices to applied to MUSIC. In [7] we developed a direction finding algorithm based on the maximal set of nonredundant fourth-order cumulants (MSNC algorithm). In [8] we proposed a Toeplitz approximation method in fourth-order cumulant domain (CTAM algorithm). The common point of the above algorithms is that the true (infinite snapshots) cumulant matrices are similar to the true noise-free covariance matrix. The differences of them, which result in different performance, rely on the way to compose the estimated cumulant matrices. We notice that these algorithms can be put into one category, that is, MUSIC algorithms in fourth-order cumulant domain (FOCD-MUSIC's).

Few FOCD-MUSIC's have been evaluated analytically. Moulines and Cardoso [5] made an asymptotic analysis of the algorithms in [3] and [5]. But the process and result of their analysis, compared to the those presented in this paper, are very complicated. Fan and Younan [9] also made a statistical analysis of the algorithm in [3]. Their analysis is based on an assumption that the sample cumulant errors due to finite data length are uncorrelated random variables with identical variance. We point out in this paper that this assumption is generally *not* true, especially when the spatial emitters are spaced closely (e.g., less than half of the beamwidth).

In this paper, we present a unifying and explicit analysis of the class of FOCD-MUSIC's. The Cramér-Rao lower bound for DOA estimation in fourth-order cumulant domain (FOCD-CRB) is also derived. Both the analysis and the derivation are based on the statistical characterization of the sample fourth-order cumulants, which is also presented in this paper.

2. Problem Formulation

Assume P narrowband plane waves are incident on an arbitrary array of M ($M>P$) sensors. The output of the mth sensor of the array can be written by

$$y_m(t) = \sum_{p=1}^{P} s_p(t) v_m(\theta_p) + w_m(t)\,,\ t=1, \dots, N, \quad (1)$$

where N is the number of snapshots, $s_p(t)$ is the complex envelope, $p = 1, \dots, P$; $w_m(t)$ is the additive noise and $v_m(\theta) = \exp\{j\vec{k}_\theta \cdot \vec{r}_m\}$, $\vec{k}_\theta$ and $\vec{r}_m$ are wave number and sensor location vectors, $m = 1, \dots, M$. We make the follow-

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ing assumptions on the array outputs. (AS1): $s_p(t)$'s are zero mean stationary non-Gaussian process with non-zero kurtosis; (AS2): $w_m(t)$'s are zero mean circular Gaussian process; (AS3): $s_p(t)$'s and $w_m(t)$'s are statistically independent themselves and of each other. Based on these assumptions and the properties of cumulants, the true second- to eighth-order cumulants can be easily derived as follows:

$$R_y(k,m) = \sum_{p=1}^{P} \gamma_{2,p} v_k(\theta_p) v_m^*(\theta_p) + R_w(k,m) \tag{2}$$

$$C_{4y}(k_1,k_2,m_1,m_2) = \sum_{p=1}^{P} \gamma_{4,p} v_{k_1}(\theta_p) v_{k_2}(\theta_p) v_{m_1}^*(\theta_p) v_{m_2}^*(\theta_p) \tag{3}$$

$$C_{6y}(k_1 \cdots k_3, m_1 \cdots m_3) = \sum_{p=1}^{P} \gamma_{6,p} v_{k_1}(\theta_p) \cdots v_{k_3}(\theta_p) v_{m_1}^*(\theta_p) \cdots v_{m_3}^*(\theta_p) \tag{4}$$

$$C_{8y}(k_1 \cdots k_4, m_1 \cdots m_4) = \sum_{p=1}^{P} \gamma_{8,p} v_{k_1}(\theta_p) \cdots v_{k_4}(\theta_p) v_{m_1}^*(\theta_p) \cdots v_{m_4}^*(\theta_p) \tag{5}$$

The sample fourth-order cumulant can be estimated by

$$\begin{aligned} \hat{C}_{4y}^N(k_1,k_2,m_1,m_2) &= \frac{1}{N} \sum_{t=1}^{N} y_{k_1}(t) y_{k_2}(t) y_{m_1}^*(t) y_{m_2}^*(t) \\ &+ \frac{1}{N^2} \sum_{t,u=1}^{N} [\, y_{k_1}(t) y_{m_1}^*(t) y_{k_2}(u) y_{m_2}^*(u) \\ &+ y_{k_1}(t) y_{m_2}^*(t) y_{k_2}(u) y_{m_1}^*(u) \,] \end{aligned} \tag{6}$$

3. Asymptotic Characterization of Sample Fourth-Order Cumulants

The sample cumulant error due to finite snapshots is defined by

$$\Delta C_{4y}^N(k_1,k_2,m_1,m_2) = \hat{C}_{4y}^N(k_1,k_2,m_1,m_2) - C_{4y}(k_1,k_2,m_1,m_2) \tag{7}$$

Theorem 1. Under (AS1)-(AS3), the estimate of fourth-order cumulant (6) is asymptotically unbiased. The asymptotic covariance of the sample cumulants is given by

$$\begin{aligned} &\lim_{N\to\infty} N \cdot \mathrm{E}\{\Delta C_{4y}^N(k_1,k_2,m_1,m_2) \Delta C_{4y}^N(k_3,k_4,m_3,m_4)\} \\ &= G_8 + G_{62} + G_{44} + G_{422} + G_{2222} \end{aligned} \tag{8a}$$

that is

$$\begin{aligned} &\mathrm{AE}\{\Delta C_{4y}^N(k_1,k_2,m_1,m_2) \Delta C_{4y}^N(k_3,k_4,m_3,m_4)\} \\ &= (G_8 + G_{62} + G_{44} + G_{422} + G_{2222}) / N \end{aligned} \tag{8b}$$

where "AE" denotes asymptotic mean; G_8, G_{62}, G_{44}, G_{422}, and G_{2222} are functions of cumulants:

$$G_8 = C_{8y}(k_1,k_2,k_3,k_4,m_1,m_2,m_3,m_4);$$

$$\begin{aligned} G_{62} = \; & C_{6y}(k_1,k_2,k_3,m_1,m_3,m_4)R_y(k_4,m_2) + C_{6y}(k_1,k_2,k_3,m_2,m_3,m_4)R_y(k_4,m_1) \\ & + C_{6y}(k_1,k_2,k_4,m_1,m_3,m_4)R_y(k_3,m_2) + C_{6y}(k_1,k_2,k_4,m_2,m_3,m_4)R_y(k_3,m_1) \\ & + C_{6y}(k_1,k_3,k_4,m_1,m_2,m_3)R_y(k_2,m_4) + C_{6y}(k_1,k_3,k_4,m_1,m_2,m_4)R_y(k_2,m_3) \\ & + C_{6y}(k_2,k_3,k_4,m_1,m_2,m_3)R_y(k_1,m_4) + C_{6y}(k_2,k_3,k_4,m_1,m_2,m_4)R_y(k_1,m_3); \end{aligned}$$

$$\begin{aligned} G_{44} = \; & C_{4y}(k_1,k_2,m_1,m_3)C_{4y}(k_3,k_4,m_2,m_4) \\ & + C_{4y}(k_1,k_2,m_1,m_4)C_{4y}(k_3,k_4,m_2,m_3) + C_{4y}(k_1,k_2,m_2,m_3)C_{4y}(k_3,k_4,m_1,m_4) \\ & + C_{4y}(k_1,k_2,m_2,m_4)C_{4y}(k_3,k_4,m_1,m_3) + C_{4y}(k_1,k_2,m_3,m_4)C_{4y}(k_3,k_4,m_1,m_2) \\ & + C_{4y}(k_1,k_3,m_1,m_2)C_{4y}(k_2,k_4,m_3,m_4) + C_{4y}(k_1,k_3,m_1,m_3)C_{4y}(k_2,k_4,m_2,m_4) \\ & + C_{4y}(k_1,k_3,m_1,m_4)C_{4y}(k_2,k_4,m_2,m_3) + C_{4y}(k_1,k_3,m_2,m_3)C_{4y}(k_2,k_4,m_1,m_4) \\ & + C_{4y}(k_1,k_3,m_2,m_4)C_{4y}(k_2,k_4,m_1,m_3) + C_{4y}(k_1,k_3,m_3,m_4)C_{4y}(k_2,k_4,m_1,m_2) \\ & + C_{4y}(k_1,k_4,m_1,m_2)C_{4y}(k_2,k_3,m_3,m_4) + C_{4y}(k_1,k_4,m_1,m_3)C_{4y}(k_2,k_3,m_2,m_4) \\ & + C_{4y}(k_1,k_4,m_1,m_4)C_{4y}(k_2,k_3,m_2,m_3) + C_{4y}(k_1,k_4,m_2,m_3)C_{4y}(k_2,k_3,m_1,m_4) \\ & + C_{4y}(k_1,k_4,m_2,m_4)C_{4y}(k_2,k_3,m_1,m_3) + C_{4y}(k_1,k_4,m_3,m_4)C_{4y}(k_2,k_3,m_1,m_2); \end{aligned}$$

$$\begin{aligned} G_{422} = \; & C_{4y}(k_3,k_4,m_1,m_2)R_y(k_1,m_3)R_y(k_2,m_4) + C_{4y}(k_3,k_4,m_1,m_2)R_y(k_1,m_4)R_y(k_2,m_3) \\ & + C_{4y}(k_1,k_2,m_3,m_4)R_y(k_3,m_1)R_y(k_4,m_2) + C_{4y}(k_1,k_2,m_3,m_4)R_y(k_3,m_2)R_y(k_4,m_1) \\ & + C_{4y}(k_1,k_3,m_1,m_3)R_y(k_2,m_4)R_y(k_4,m_2) + C_{4y}(k_1,k_3,m_2,m_4)R_y(k_2,m_3)R_y(k_4,m_1) \\ & + C_{4y}(k_1,k_3,m_1,m_4)R_y(k_2,m_3)R_y(k_4,m_2) + C_{4y}(k_1,k_3,m_2,m_3)R_y(k_2,m_4)R_y(k_4,m_1) \\ & + C_{4y}(k_2,k_4,m_1,m_3)R_y(k_1,m_4)R_y(k_3,m_2) + C_{4y}(k_2,k_4,m_2,m_4)R_y(k_1,m_3)R_y(k_3,m_1) \\ & + C_{4y}(k_2,k_4,m_1,m_4)R_y(k_1,m_3)R_y(k_3,m_2) + C_{4y}(k_2,k_4,m_2,m_3)R_y(k_1,m_4)R_y(k_3,m_1) \\ & + C_{4y}(k_1,k_4,m_1,m_3)R_y(k_2,m_4)R_y(k_3,m_2) + C_{4y}(k_1,k_4,m_2,m_4)R_y(k_2,m_3)R_y(k_3,m_1) \\ & + C_{4y}(k_1,k_4,m_1,m_4)R_y(k_2,m_3)R_y(k_3,m_2) + C_{4y}(k_1,k_4,m_2,m_3)R_y(k_2,m_4)R_y(k_3,m_1) \\ & + C_{4y}(k_2,k_3,m_1,m_3)R_y(k_1,m_4)R_y(k_4,m_2) + C_{4y}(k_2,k_3,m_2,m_4)R_y(k_1,m_3)R_y(k_4,m_1) \\ & + C_{4y}(k_2,k_3,m_1,m_4)R_y(k_1,m_3)R_y(k_4,m_2) + C_{4y}(k_2,k_3,m_2,m_3)R_y(k_1,m_4)R_y(k_4,m_1); \end{aligned}$$

$$\begin{aligned} G_{2222} = \; & R_y(k_3,m_1)R_y(k_4,m_2)R_y(k_1,m_3)R_y(k_2,m_4) + R_y(k_3,m_1)R_y(k_4,m_2)R_y(k_2,m_3)R_y(k_1,m_4) \\ & + R_y(k_4,m_1)R_y(k_3,m_2)R_y(k_1,m_3)R_y(k_2,m_4) + R_y(k_4,m_1)R_y(k_3,m_2)R_y(k_2,m_3)R_y(k_1,m_4) \end{aligned}$$

where $R_y(...)$, $C_{4y}(...)$, $C_{6y}(...)$ and $C_{8y}(...)$ are second- to eighth-order cumulants of $y(t)$, as shown in (2)-(5).

Proof: The derivation of (8) is somewhat lengthy but it is straightforward by using (6) and the moment-to-cumulant formulas [1]. ∎

Remark: The statistical characterization of sample fourth-order cumulants of a *single realization* of random process was discussed in [10] and [11]. In [4] an approximate expression for the covariance of the sample fourth-order cumulants of *multiple realizations (snapshots)* of the process used in array processing was derived. But the expression is a function of higher-order moments. The result presented in this paper is a function of higher-order cumulants and is convenient for computing in most applications such as array processing.

It is seen form the theorem that the sample cumulants can not generally be regarded as uncorrelated and equivariace random variables. However, the joint distribution of sample cumulants is asymptotically Gaussian (not circular in general)[11, 12].

4. Asymptotic Performance of FOCD-MUSIC

Let $\hat{C}$ be the estimated cumulant matrix used in FOCD-MUSIC. The true cumulant matrix has a general form of

$$C = B\Sigma B^H \tag{9}$$

where

$$\Sigma = \text{diag}[\gamma_{4,1}, \cdots, \gamma_{4,P}], \tag{10}$$

$$\gamma_{4,p} = \text{Cum}\{ s_p(t), s_p(t), s_p^*(t), s_p^*(t) \} \tag{11}$$

$$B = [b(\theta_1), \cdots, b(\theta_P)] \tag{12}$$

$$b(\theta) = [b_1(\theta), \cdots, b_{M_C}(\theta)]^T. \tag{13}$$

$b(\theta)$ is the steering vector in cumulant domain and M_c is the dimension of the cumulant matrix, which depend on the cumulant matrix specified by the algorithm.

Theorem 2. The FOCD-MUSIC DOA estimator is asymptotically unbiased. The asymptotic variance of $\hat{\theta}_p$ is given by

$$\text{AE}\{ (\Delta\theta_p)^2 \} = 2\,\text{Re}\Bigg[\sum_{q_1,q_2=1}^{P} \sum_{\substack{i_1,i_2=1 \\ i_1 \neq q_1, i_2 \neq q_2}}^{M_c} \frac{(u_{i_1} \otimes u_{i_2})^H \,\text{AE}\{ \Delta C \otimes \Delta C \} (u_{q_1} \otimes u_{q_2})}{(\lambda_{i_1} - \lambda_{q_1})(\lambda_{i_2} - \lambda_{q_2})} \cdot G_{p,q_1}^T u_{i_1} u_{i_2}^T G_{p,q_2} + \frac{(u_{i_1} \otimes u_{i_2}^*)^H \,\text{AE}\{ \Delta C \otimes \Delta C^* \} (u_{q_1} \otimes u_{q_2}^*)}{(\lambda_{i_1} - \lambda_{q_1})(\lambda_{i_2} - \lambda_{q_2})} \cdot G_{p,q_1}^T u_{i_1} u_{i_2}^H G_{p,q_2}^* \Bigg] \Big/ \ddot{S}^2(\theta_p, U_s) \tag{14}$$

where $\Delta\theta_p = \hat{\theta}_p - \theta_p$, $\otimes$ denotes Kronecker product, λ_i's and u_i's are eigenvalues and the corresponding normalized eigenvectors of C, with $|\lambda_i|$ in decreasing order ($\lambda_i = 0$ for $i = P+1, \ldots, M$), $U_s = [u_1, \cdots, u_P]$, and

$$\Delta C = \hat{C} - C, \; d(\theta_p) = \dot{b}(\theta_p) \tag{15}$$

$$G_{p,q} = -d^*(\theta_p) u_q^H b(\theta_p) - b^*(\theta_p) u_q^H d(\theta_p) \tag{16}$$

$$\ddot{S}(\theta_p, E_s) = 2 d^H(\theta_p)(I_{M_c} - U_s U_s^H) d(\theta_p) \tag{17}$$

Proof: The derivation of (14) require some approximations on the perturbed signal eigenvectors [13]. The details of the derivation will be given in another paper[8] and are omitted here due to space limitation. ∎

By using theorem 1 the computation of (14) is straightforward when an algorithm of FOCD-MUSIC is specified.

5. Cramér-Rao Bound in Fourth-Order Cumulant Domain

The asymptotic Cramér-Rao lower bound for cumulant-based DOA estimation may be determined in cumulant domain since the sample cumulants are asymptotically Gaussian distributed [12]. We choose all of the nonredundant sample cumulants as the "observations" in cumulant domain without loss of information.

Theorem 3. The maximal set of nonredundant sample cumulants is given by $\mathcal{C}$:

$$\mathcal{C} = \{ \hat{C}_{4y}^N(k_1, k_2, m_1, m_2) | (k_1, k_2, m_1, m_2) \in \mathcal{T} \} \tag{18}$$

$$\mathcal{T} = \{ (k_1, k_2, m_1, m_2) | k_1, k_2, m_1, m_2 \text{ satisfy CONDS} \} \tag{19}$$

where the CONDS is given by:

$$\begin{cases} k_1,\, m_1 \in [1, M];\; k_2 \in [1, k_1];\; m_2 \in [1, m_1] \\ (m_1 - 1)m_1 + 2m_2 \le (k_1 - 1)k_1 + 2k_2 \end{cases} \tag{20}$$

the number of elements in $\mathcal{C}$ is Q,

$$Q = M(M+1)(M^2 + M + 2)/8. \tag{21}$$

Proof: The result can be obtained by examining the symmetry of the fourth-order cumulants. (20) is a revised version of that in [7]. See [7] for more details. ∎

Let $\tau_q = (k_1^{(q)}, k_2^{(q)}, m_1^{(q)}, m_2^{(q)}) \in \mathcal{T}$. We stack the nonredundant sample cumulants in a vector $\hat{\xi}$

$$\hat{\xi} = [\hat{C}_{4y}^N(\tau_1), \cdots, \hat{C}_{4y}^N(\tau_Q)]^T = \hat{\xi}_R + j\hat{\xi}_I \tag{22}$$

Since the sample cumulants are not circular Gaussian in general, we create a $2Q$-dimensional real Gaussian vector $\hat{\zeta}$, $\hat{\zeta} = [\hat{\xi}_R^T, \hat{\xi}_I^T]^T$. In the case of white additive Gaussian noise, parameters to be estimated in cumulant domain are: DOAs $\theta = \{\theta_p\}$, signal cumulant $\gamma = \{\gamma_{2,p}, \gamma_{4,p}, \gamma_{6,p}, \gamma_{8,p}\}$, $p = 1, \ldots, P$, and noise power σ_w^2. Then the Fisher information matrix (FIM) [14] is given by

$$J = \begin{bmatrix} J^{\theta,\theta} & J^{\theta,\gamma} & J^{\theta,\sigma_w^2} \\ (J^{\theta,\gamma})^T & J^{\gamma,\gamma} & J^{\gamma,\sigma_w^2} \\ (J^{\theta,\sigma_w^2})^T & (J^{\gamma,\sigma_w^2})^T & J^{\sigma_w^2,\sigma_w^2} \end{bmatrix} \tag{23}$$

where

$$J_{i,k}^{\theta,\theta} = \frac{\partial \zeta^T}{\partial \theta_i} R_\zeta^{-1} \frac{\partial \zeta}{\partial \theta_k} + \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \theta_i} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \theta_k} \} \tag{24}$$

$$J_{i,k}^{\gamma,\gamma} = \frac{\partial \zeta^T}{\partial \gamma_i} R_\zeta^{-1} \frac{\partial \zeta}{\partial \gamma_k} + \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \gamma_i} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \gamma_k} \} \tag{25}$$

$$J^{\sigma_w^2,\sigma_w^2} = \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \sigma_w^2} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \sigma_w^2} \} \tag{26}$$

$$J_{i,k}^{\theta,\gamma} = \frac{\partial \zeta^T}{\partial \theta_i} R_\zeta^{-1} \frac{\partial \zeta}{\partial \gamma_k} + \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \theta_i} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \gamma_k} \} \tag{27}$$

$$J_i^{\theta,\sigma_w^2} = \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \theta_i} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \sigma_w^2} \} \tag{28}$$

$$J_i^{\gamma,\sigma_w^2} = \frac{1}{2} \text{tr}\{ R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \gamma_i} R_\zeta^{-1} \frac{\partial R_\zeta}{\partial \sigma_w^2} \}, \tag{29}$$

where $J_{i,k}^{\theta,\theta}$ is the element at *i*th row and *k*th column of $J^{\theta,\theta}$, and so on. It is easy to find that

$$R_{\zeta} = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{R_{\xi_1} + R_{\xi_2}\} & -\mathrm{Im}\{R_{\xi_1} - R_{\xi_2}\} \\ \mathrm{Im}\{R_{\xi_1} + R_{\xi_2}\} & \mathrm{Re}\{R_{\xi_1} - R_{\xi_2}\} \end{bmatrix}, \quad (30)$$

where

$$R_{\xi_1} = \mathrm{AE}\{\Delta\xi\Delta\xi^H\} = [\mathrm{AE}\{\Delta C_{4y}^N(\tau_{q_1})(\Delta C_{4y}^N(\tau_{q_2}))^*\}]_{Q\times Q} \quad (31)$$

$$R_{\xi_2} = \mathrm{AE}\{\Delta\xi\Delta\xi^T\} = [\mathrm{AE}\{\Delta C_{4y}^N(\tau_{q_1})\Delta C_{4y}^N(\tau_{q_2})\}]_{Q\times Q} \quad (32)$$

Finally, we get the asymptotic Cramér-Rao lower bound for DOA estimation in fourth-order cumulant domain (FOCD-CRB):

$$\mathrm{AE}\{\Delta\theta_p^2\}_{\text{FOCD-CRB}} = J_{p,p}^{-1}, \; p = 1,\ldots,P. \quad (33)$$

where $J_{p,p}^{-1}$ represents the element at *p*th row and column of J^{-1}. Using the result of theorem 1 the FOCD-CRB can be evaluated numerically.

6. Simulations

Simulation experiments were performed on a uniform linear array (ULA) separated by half a wavelength of the narrowband signals. Two emitters (*P*=2) broadcast BPSK waveforms from θ_1 =10° and θ_2 =10° with respect to the broadside of the array. The additive noise is white Gaussian. Three algorithms within the class of FOCD-MUSIC, named MUSIC-like[4], MSNC[7] and CTAM[8], were evaluated and compared to the conventional MUSIC[2] and FOCD-CRB by numerical theory results and Monte Carlo simulations. Each of these simulations is based on 100 independent runs. Shown in Fig. 1 are the estimation mean-square-errors (MSE's) of θ_1 versus signal-to-noise ratios (SNR's). The experiment condition is: number of sensors M = 4; number of snapshots N = 500. The close agreement between theoretical predictions and simulation results is clearly evident when the SNR is above 5 dB, and good for CTAM algorithm [8] even when SNR is as low as -5 dB. The MSNC algorithm [7] is almost of no difference with the MUSIC-like algorithm [4] while the former is less computationally expensive than the latter. It can be seen that the FOCD-MUSIC's outperform the conventional MUSIC [2] only when the SNR is somewhat low (less that 10 dB approximately). Fig. 2 and Fig. 3 plot the MSE's versus the number of snapshots (*N*) while the number of sensors (*M*) varies from 4 in Fig. 2 to 6 in Fig. 3 (which means the relative spatial separation between the two emitters is increased). Again, the analytical predictions compare favorably with the simulation results when the number of snapshots is not quite low (greater than 100 approximately). It is shown that the FOCD-MUSIC's outperform the conventional MUSIC when spatial emitters are rather closely spaced (less than half of the beamwidth approximately). This is also shown by Fig. 4, where the MSE's are versus the variation of the difference of spatial angles of the two emitters.

7. Conclusions

A unifying asymptotic analysis of FOCD-MUSIC was presented and the FOCD-CRB was also derived. Simulation experiments validated the analytical results. Three typical FOCD-MUSIC's were evaluated and compared to covariance-based MUSIC and FOCD-CRB. The MUSIC-like and MSNC have an almost identical performance as predicted in [7]. CTAM behaves rather robustnessly in all scenarios. The FOCD-MUSIC's are not superior to the conventional MUSIC algorithm in all cases. Nevertheless, they outperform the conventional MUSIC with reduced variances and improved robustness when the spatial sources are closely spaced and the signal-to-noise ratios (SNRs) are relatively low. The FOCD-MUSIC's are inefficient. It can be seen from Fig. 2 and Fig. 3 that the estimation covariances decrease parallel with the FOCD-CRB while the number of snapshots goes large, which means that the FOCD-MUSIC's are not asymptotically efficient.

References

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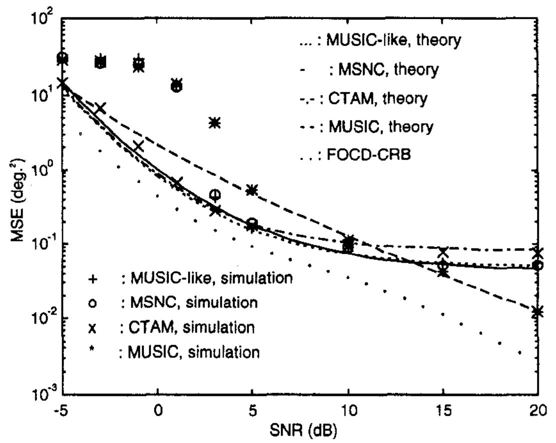

Fig. 1 Mean-square-errors (MSE's) versus signal-to-noise ratios (SNR's) for $\theta_1 = 10°$. $M = 4$, N=500.

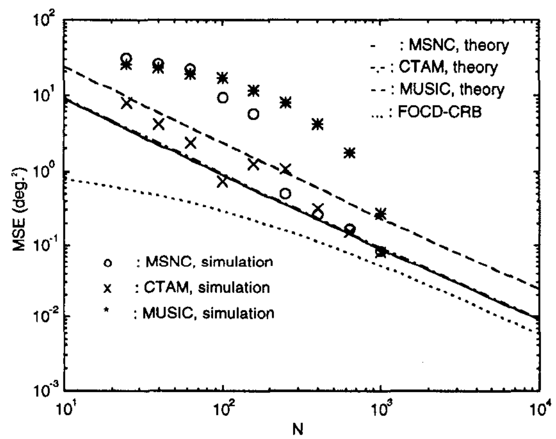

Fig. 2 Mean-square-errors (MSE's) versus number of snapshots (N) for $\theta_1 = 10°$. $M = 4$, SNR= 5 dB.

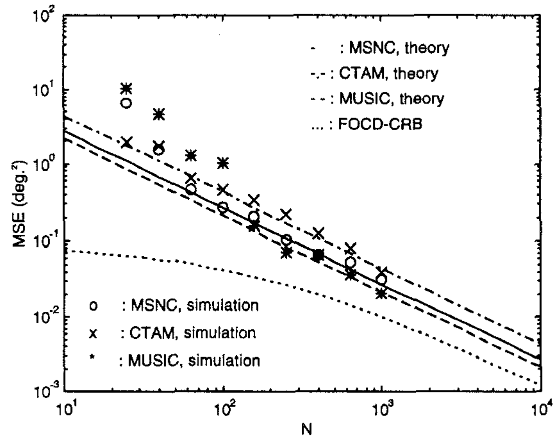

Fig. 3 Mean-square-errors (MSE's) versus number of snapshots (N) for $\theta_1 = 10°$. $M = 6$, SNR = 5 dB.

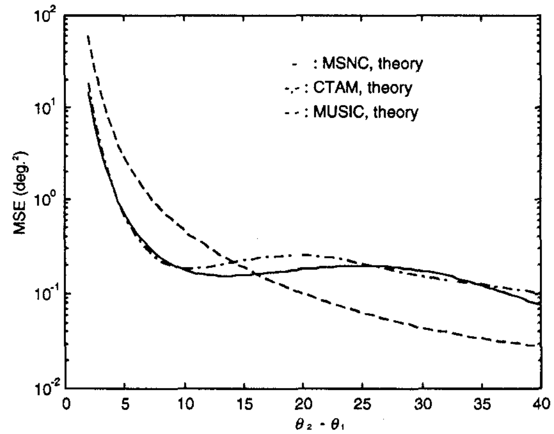

Fig. 4 Mean-square-errors (MSE's) versus the angle differences of the two emitters for $\theta_1 = 10°$. $M = 4$, N =500, SNR =5 dB.