Summary of Estimators

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Approaches to Estimation

- Classical Approach
  - Unknown parameter vector **θ** -***Deterministic Constant***
    - – Cramer - Rao Lower Bound
    - – Rao-Blackwell-Lehmann- Scheffe
    - – Best Linear Unbiased Estimator
    - – Maximum Likelihood Estimator
    - – Least Squares Estimator
    - – Method of Moments

- Bayesian Approach
  - Known parameter vector **θ** –***Realization of a Random Vector***
    - – Minimum Mean Square Error Estimator
    - – Maximum A Posteriori Estimator
    - – Linear Minimum Mean Square Error Estimator

Estimator Overview

- Data Models
- Estimator
- Optimality/Error Criterion
- Performance
- Comments etc..

Cramer Rao Lower Bound(CRLB)

a. Data Model/Assumption

PDF p(**x;θ**) is known.

b. Estimator

If the equality condition for the CRLB

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = I(\boldsymbol{\theta})(\mathbf{g}(\mathbf{x}) - \boldsymbol{\theta})$$

is satisfied, then the estimator is $\hat{\boldsymbol{\theta}} = \mathbf{g}(\mathbf{x})$

where **I(θ)** is a p x p matrix dependent only on **θ** and **g(x)** is a p-D function of data x.

c. Optimality/Criterion

$\hat{\theta}$ **achieves the CRLB**, the lower bound on the variance for any unbiased estimator(and hence is said to be efficient), and is therefore the MVU estimator.

d. Performance

It is **unbiased** and has the **minimum variance**

$$var(\hat{\theta}_i) = [\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii} \quad i=1,2,.....p$$

where

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = E\left[\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \theta_j}\right]$$

e. Comments

An efficient **Estimator may not exist** & this approach may fail

Rao-Blackwell-Lehmann-Scheffe(RBLS)

a. Data Model/Assumption

PDF p($\mathbf{x};\boldsymbol{\theta}$) is known.

b. Estimator

i. Find a *sufficient statistic* **T(x)** by factoring PDF as

$$p(\mathbf{x};\boldsymbol{\theta})=g(\mathbf{T}(\mathbf{x}),\boldsymbol{\theta})h(\mathbf{x})$$

ii. If E[**T(x)**]=$\boldsymbol{\theta}$,then $\hat{\boldsymbol{\theta}}$ =**T**(**x**).If not, we must find a p-dimensional function **g** so that E[**g**(**T**)]= $\boldsymbol{\theta}$, and then $\hat{\boldsymbol{\theta}}$ = **g(T).**

c. Optimality/Criterion

$\hat{\boldsymbol{\theta}}$ is the MVU estimator.

d. Performance

Unbiased but **variance depends upon PDF** -no general formula is available

e. Comments

"**Completeness**" of sufficient statistic must be checked. A p-dimensional sufficient statistic may not exist so this method may fail.

Best Linear Unbiased Estimator

a. Data Model/Assumptions

$$E(\mathbf{x})=\mathbf{H\theta}$$

where **H** is an N x p (N>p) known matrix and **C**, the covariance matrix of **x**, is known. Equivalently, we have

$$\mathbf{x} = \mathbf{H\theta} + \mathbf{w}$$

where $E(\mathbf{w})=0$ and $\mathbf{C_w} = \mathbf{C}$.

b. Estimator

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$$

c. Optimality/Error Criterion

$\hat{\theta}_i$ for i = 1,2,….,p has **minimum variance** of all unbiased estimators that are *linear* in **x.**

d. Performance

$\hat{\theta}_i$ for i = 1,2,…,p is unbiased. The variance is

$$\mathrm{var}(\hat{\theta}_i) = [(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}]_{ii} \qquad i = 1,2,\ldots\ldots,p$$

e. Comments

If **w** is a Gaussian RV ,then is $\hat{\boldsymbol{\theta}}$ also the ***MVU estimator*** (for all linear & non linear functions of **x**).

Maximum Likelihood Estimator

a. Data Model/Assumptions

PDF p(**x;θ**) is known

b. Estimator

$\hat{\boldsymbol{\theta}}$ is the value of **θ** maximizing p(**x;θ**), where **x** is replaced by the observed data samples.

c. Optimality/Error Criterion

Not optimal in general. Under certain conditions on the PDF, however, the MLE is efficient for large data records or as N →∞ (asymptotically).Hence, asymptotically it is the MVU estimator.

d. Performance

For finite N depends on PDF-no general formula is available. Asymptotically, under certain conditions

$$\hat{\theta} \sim N(\theta, \mathrm{I}^{-1}(\theta))$$

e. Comments

If an MVU estimator exists, the MLE procedure will produce it .

Least Squares Estimator

a. Data Model/Assumption

$x[n]=s[n;\boldsymbol{\theta}] + w[n] \quad n=0,1,\ldots\ldots,N-1$

where the signal $s[n;\boldsymbol{\theta}]$ depends explicitly on the unknown parameters. Equivalently, the model is

$$\mathbf{x}=\mathbf{s}(\boldsymbol{\theta}) + \mathbf{w}$$

where **s** is a known N-dimensional function of **θ** and the noise or perturbation **w** has zero mean.

b. Estimator

$\hat{\theta}$ is the value that minimizes

$$J(\boldsymbol{\theta}) = (\mathbf{x}-\mathbf{s}(\boldsymbol{\theta}))^T(\mathbf{x}-\mathbf{s}(\boldsymbol{\theta})) = \sum_{n=0}^{N-1}(\mathbf{x}[n] - s[n;\boldsymbol{\theta}])^2])$$

c. Optimality/Error Criterion

None In General

d. Performance

Depends on PDF of **w**-no general formula is available

e. Comments

The fact that we are ***minimizing a LS error minimizing a LS error criterion does not in general translate into minimizing the estimation error***. Also **w** is a Gaussian RV $N(0,\sigma^2\mathbf{I})$, then the LSE is equivalent to MLE.

Method of Moments

a. Data Model/Assumptions

There are p moments μ_i=E(x^i[n]) for i = 1,2, . . . ,p, which depends upon **θ** in a known way. *The entire PDF need not be known*.

b. Estimator

If **μ = h(θ),** where **h** is an invertible p-D function of **θ** & **μ**=[μ_1 μ_2 . . . μ_p]^T, then

$$\hat{\theta} = \mathrm{h}^{-1}(\hat{\mu})$$

c. Optimality/Error Criterion

None in General

d. Performance

For finite N it depends on PDF of **x**. However for large data records (asymptotically), if

$\hat{\theta}_i = g_i(\hat{\boldsymbol{\mu}})$ then,

$$E(\hat{\theta}_i) = g_i(\boldsymbol{\mu})$$

$$var(\hat{\theta}_i) = \left(\frac{\partial\, g_i}{\partial \hat{\boldsymbol{\mu}}}\right)^T_{\hat{\mu}=\mu} \mathrm{C}_{\hat{\mu}} \left(\frac{\partial\, g_i}{\partial \hat{\boldsymbol{\mu}}}\right)_{\hat{\mu}=\mu} \quad \text{for i = 1,2 … ,p}$$

e. Comments

Usually very ***easy to implement***.

Minimum Mean Square Error Estimator

a. *Data model/Assumptions*

The joint PDF p($\mathbf{x},\boldsymbol{\theta}$) is known, where $\boldsymbol{\theta}$ is considered to be a RV. Usually, p($\mathbf{x}|\boldsymbol{\theta}$) is specified as the data model and p($\boldsymbol{\theta}$) as the prior PDF for $\boldsymbol{\theta}$, so that p($\mathbf{x},\boldsymbol{\theta}$)=p($\mathbf{x}|\boldsymbol{\theta}$)p($\boldsymbol{\theta}$).

b. Estimator

$\hat{\theta}$ = E($\boldsymbol{\theta}|\mathbf{x}$) ,where the expectation represents the posterior PDF

$$p(\boldsymbol{\theta}|\mathbf{x}) = \frac{p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})\,d\boldsymbol{\theta}}$$

If $\mathbf{x},\boldsymbol{\theta}$ are jointly Gaussian, $\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}(\mathbf{x} - E(\mathbf{x}))$

c. Optimality/Error criterion

$\hat{\theta}_i$ minimizes the Bayesian MSE

$\mathrm{Bmse}(\hat{\theta}_i)=E[(\theta_i-\hat{\theta}_i)^2]$ i=1 ,2 , . . , p where the Expectation is w.r.t p($\mathbf{x}$, θ_i)

d. Performance

The error $\varepsilon_i = \theta_i - \hat{\theta}_i$ has zero mean & variance

$$\mathrm{var}(\varepsilon_i) = Bmse(\hat{\theta}_i) = \int [\mathbf{C}_{\theta|x}]_{ii}\, p(\mathbf{x})d\mathbf{x}$$

If $\mathbf{x}$, $\boldsymbol{\theta}$ are jointly Gaussian, then the error is Gaussian with zero mean & variance

$\mathrm{var}(\varepsilon_i)=[\mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}]_{ii}$

e. Comments

In the non-Gaussian case, this will be difficult to implement.

Maximum A Posteriori Estimator

a. *Data Model/Assumptions*

Same as for the MMSE estimator.

b. *Estimator*

$\hat{\theta}$ is the value of **θ** that maximizes p(**θ|x**) or, equivalently the value that maximizes p(**x|θ**)p(**θ**). If **x,θ** are jointly Gaussian, $\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\boldsymbol{\theta}\mathbf{x}}\mathbf{C}_{\mathbf{xx}}^{-1}(\mathbf{x} - E(\mathbf{x}))$

c. *Optimality/Error Criterion*

Minimizes the ***"hit or miss"*** cost function.

d. *Performance*

Depends on PDF-no general formula available. If **x,θ** are jointly Gaussian, then the performance is identical to that of MMSE estimator.

e. *Comments*

For PDF's whose ***mean & mode are the same***, the MMSE & MAP estimators will be identical, i.e., the Gaussian PDF, for example.

Linear Minimum Mean Square Error Estimator

a. Data Model/Assumptions

The first two moments of the joint PDF p(x,θ) are known or the mean and covariance

$$\begin{bmatrix} E(\boldsymbol{\theta}) \\ E(\boldsymbol{x}) \end{bmatrix} \qquad \begin{pmatrix} C_{\theta\theta} & C_{\theta x} \\ C_{x\theta} & C_{xx} \end{pmatrix}$$

b*. Estimator*

$$\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}(\mathbf{x} - E(\mathbf{x}))$$

c. Optimality/Error Criterion

$\hat{\theta}_i$ has the minimum Bayesian MSE of all estimators that are *linear* function of x

d. Performance

The error $\varepsilon_i = \theta_i - \hat{\theta}_i$ has zero mean & variance

$$\mathrm{var}(\varepsilon_i) = \mathrm{Bmse}(\hat{\theta}_i) = [\mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}]_{ii}$$

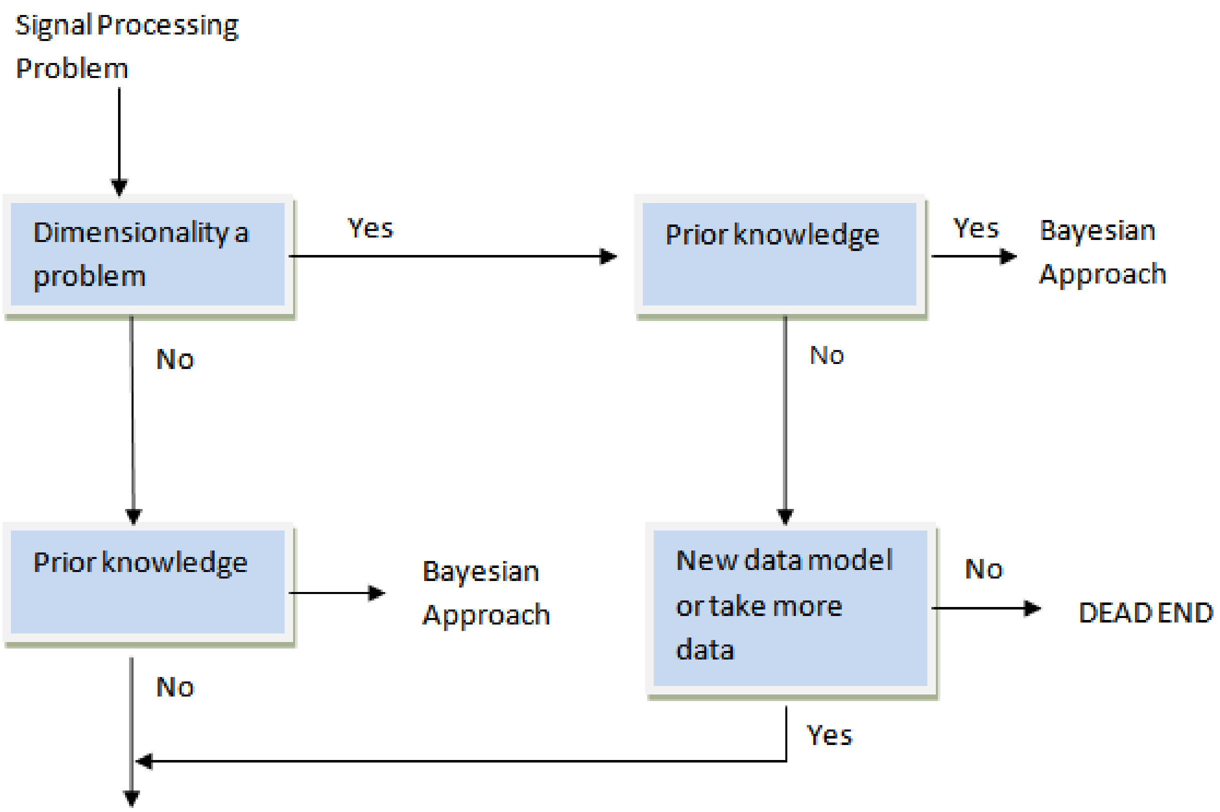
e. Comments

If **x,θ** are jointly Gaussian, this is identical to the MMSE & MAP estimators.

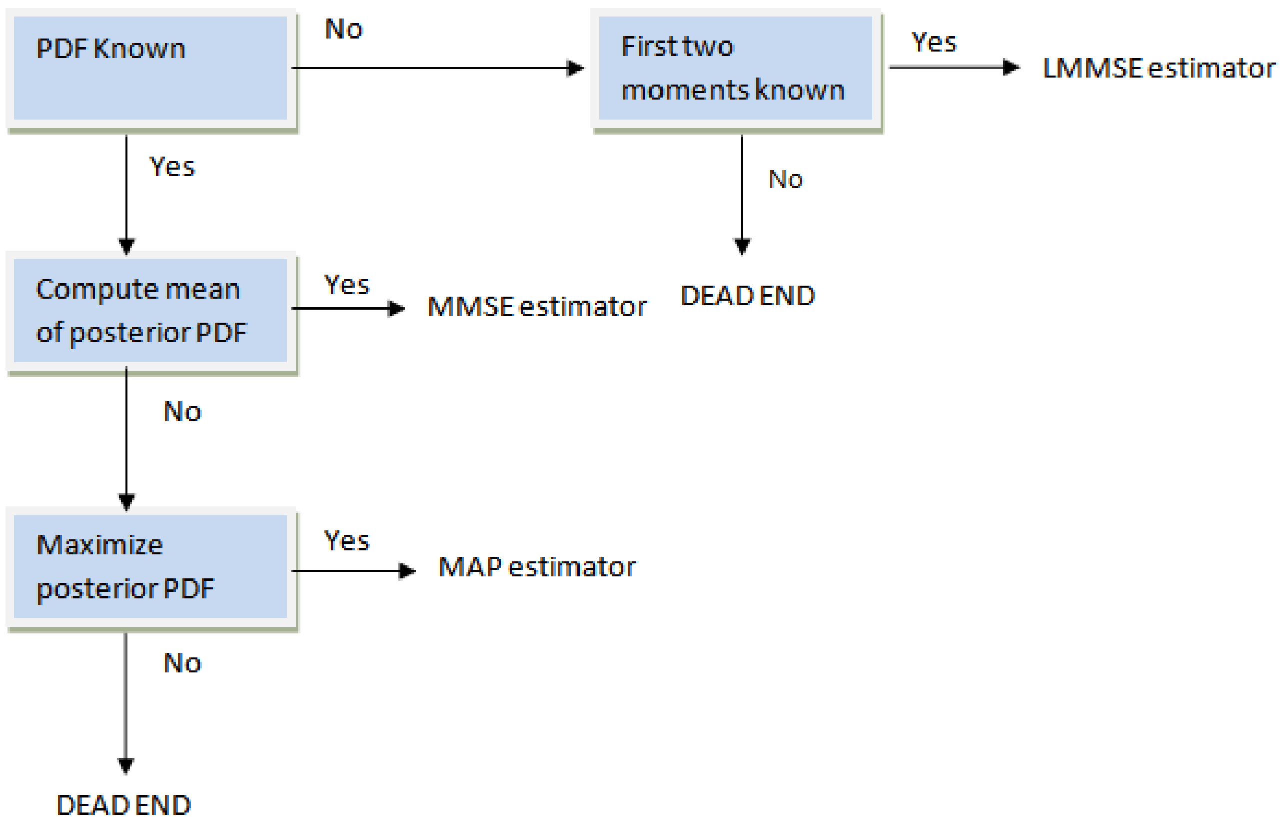
Choosing an Estimator

- The Choice of an estimator should begin with the search for an optimal estimator that is computationally feasible.
- If the search proves futile, then suboptimal estimators should be investigated.
- Even if the estimator is optimal for the given data model, its performance may not be adequate. In such a case, the data model may need to be modified.

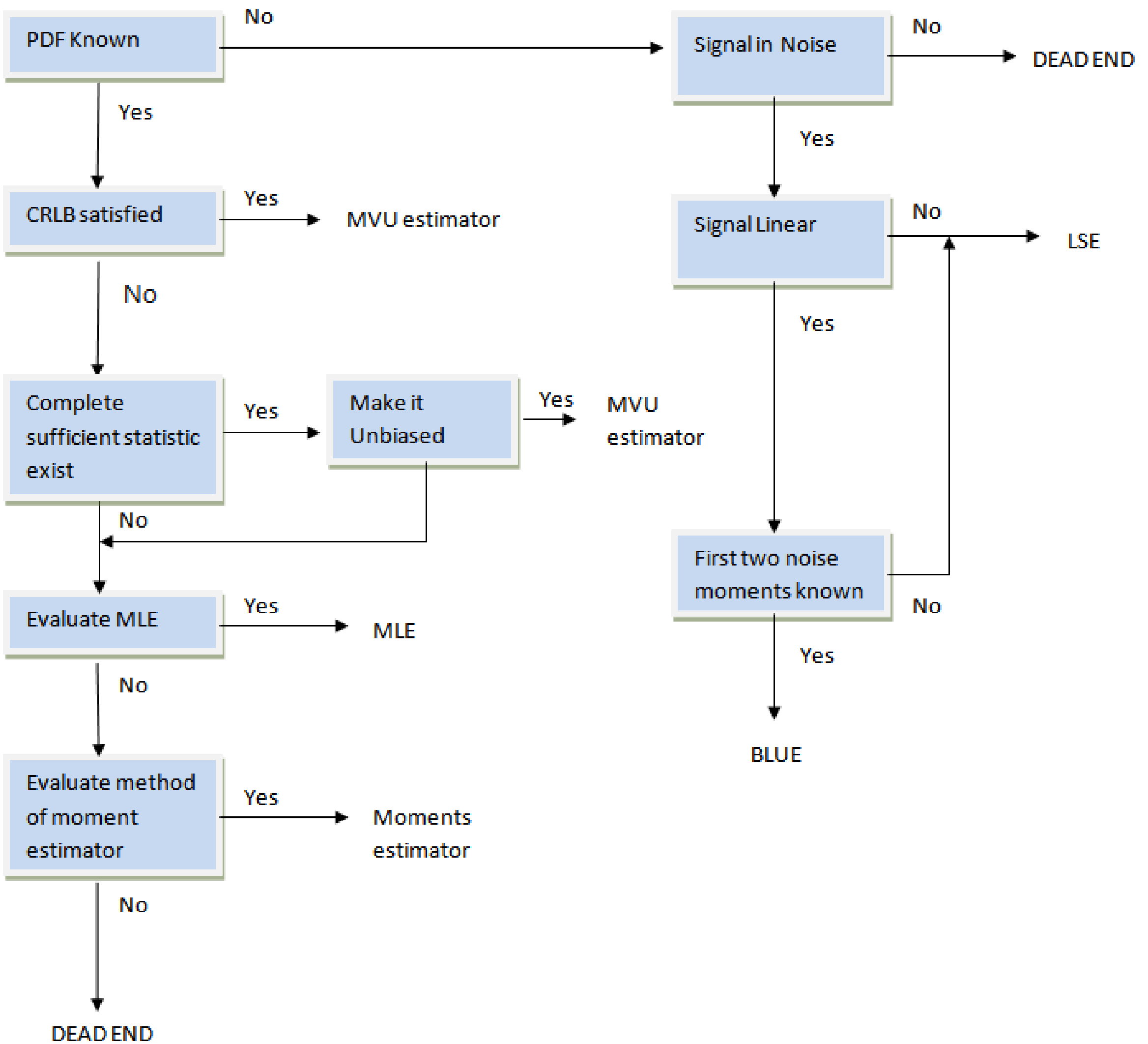
Classic v/s Bayesian



Bayesian Approach



Classical Approach



Final Comments

- Selection of Good Data Model
  - Complex enough to describe principle features
  - Simple enough to allow an estimator that is optimal & easily feasible.
- Determination of optimal estimators not possible
  - Search of an MVU estimator in classical approach
- Implementation of optimal estimator not possible
  - MMSE estimator in Bayesian estimation