KALMAN FILTERING

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Outline:

- Introduction
- Dynamic signal models
    - a)Basic Model
    - b)Gauss-Markov Process
- Kalman filter design
- Properties of kalman filter

Introduction:

- Filter is need to reduce the amount of noise present in a signal by comparison with an estimation of desired noise less signals.
- Here We are designing the filter by using statistical methods not by spectral component basis.

  i.e Finding the coefficients of filter using statistical approach

Dynamic signal models

Basic Model

let $x[n]= A+ w[n]$

$\rightarrow$ A is the parameter to be estimated.

$\rightarrow$ w[n] is a WGN.

But in Dynamic model

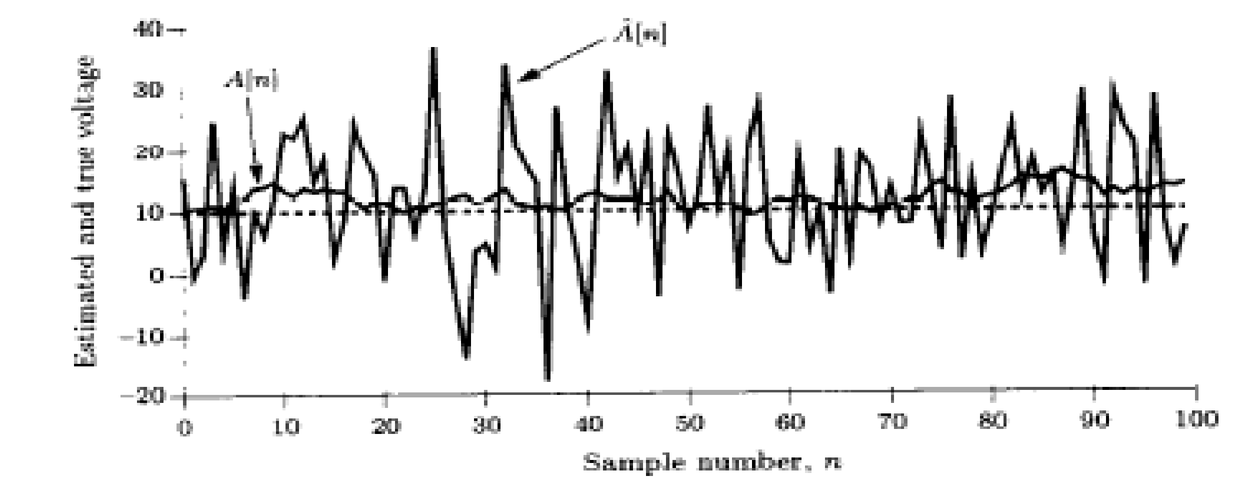
$x[n]=A[n]+w[n]$

where A[n] is the true voltage at time 'n' .

i.e A[n] is a function of time.

X[n] is a measurement which we know.

True voltage and estimator Â[n]



Let Estimator $\hat{A}[n] = x[n]$

$E(\hat{A}[n]) = E(x[n]) = A[n]$

$var(\hat{A}[n]) = E(x[n]) = \sigma^2$

Estimator will be inaccurate due to lack of averaging

- From true value we can Identify ,the successive samples of A[n] will not be too different i.e they are correlated.
- Thus the imposition of a correlation constraint will prevent the estimate of A[n] from fluctuating too widely in time.

Hence by comparing successive measurements

We can get the information of innovation

Gauss Markov process

$s[n] = a\, s[n-1]+u[n] \quad \text{for } n \geq 0$

$u[n]$ is WGN with variance σ_u^2

$s[-1] \sim N(\mu_s, \sigma_s^2)$

$s[0]= a\, s[-1]+u[0]$

$s[1]= a\, s[0]+u[1]$

$= a^2 s[-1]+a\, u[0]+u[1]$

In general we have

$$s[n] = a^{n+1} s[-1]+ \sum_{k=-\infty}^{\infty} a^k u[n-k]$$

$$
\begin{aligned}
E(s[n]) &= a^{n+1}E(s[-1]) \\
&= a^{n+1}\mu_s.
\end{aligned}
$$

Covariance between samples s[m] and s[n]

$$
\begin{aligned}
c_s[m,n] &= E\left[(s[m]-E(s[m]))(s[n]-E(s[n]))\right] \\
&= E\left[\left(a^{m+1}(s[-1]-\mu_s)+\sum_{k=0}^{m}a^k u[m-k]\right)\right. \\
&\qquad \left.\cdot\left(a^{n+1}(s[-1]-\mu_s)+\sum_{l=0}^{n}a^l u[n-l]\right)\right] \\
&= a^{m+n+2}\sigma_s^2+\sum_{k=0}^{m}\sum_{l=0}^{n}a^{k+l}E(u[m-k]u[n-l]).
\end{aligned}
$$

But

$$
E(u[m-k]u[n-l]) = \sigma_u^2\delta[l-(n-m+k)]
$$

or

$$
E(u[m-k]u[n-l]) = \begin{cases} \sigma_u^2 & l=n-m+k \\ 0 & \text{otherwise.} \end{cases}
$$

- $Var(s[n]) = C_s[n,n]$

$$= a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=-\infty}^{\infty} a^{2k}$$

s[n] is not WSS since the mean depends on 'n' and covariance depends on m and n, not the difference.

As $n \to \infty$

$E[s[n]] = 0;$

$C_s[m,n] = \sigma_u^2(a^{m-n})/(1-a^2)$

Thus $r_{ss}[k] = C_s[m,n]_{m-n=k} = \sigma_u^2(a^k)/(1-a^2)$ for $k \geq 0$

In Gauss-Markov process the mean and variance can also be expressed recursively.

$$E[s[n]] = a\,E[s[n-1]]+E[u[n]]$$

$$E[s[n]] = a\,E[s[n-1]] \quad \text{since } E[u[n]]=0;$$

$$\begin{aligned}\mathrm{var}(s[n]) &= E[(s[n]-E(s[n]))^2] \\ &= E[(a\,s[n-1]+u[n]-a\,E(s[n-1]))^2] \\ &= a\,\mathrm{var}(s[n-1])+\sigma_u^2\end{aligned}$$

Since $s[n-1]$ depends only on $\{s[-1],u[0]\ldots.u[n-1]\}$ and independent of $u[n]$ by assumption.

$$E(u[n]s[n-1])=0$$

Pth order Gauss-Markov Process

$$S[n] = -\sum_{k=1}^{p} a[k]\, s[n-k] + u[n]$$

Here S[n] depends on 'p' previous samples

Define a vector **s**[n-1]

$$\mathbf{s}[n-1] = \begin{bmatrix} s[n-p] \\ s[n-p+1] \\ \vdots \\ s[n-1] \end{bmatrix}.$$

$$\begin{bmatrix} s[n-p+1] \\ s[n-p+2] \\ \vdots \\ s[n-1] \\ s[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a[p] & -a[p-1] & -a[p-2] & \cdots & -a[1] \end{bmatrix}}_{A} \begin{bmatrix} s[n-p] \\ s[n-p+1] \\ \vdots \\ s[n-2] \\ s[n-1] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u[n]$$

This can be rewrite in the form

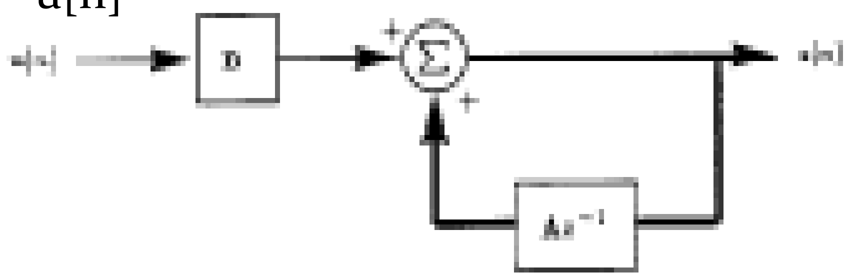
$s[n]= A\, s[n-1] + B\, u[n] \quad n \geq 0$

Multi input –Multi output model

Equivalent vector model

B

u[n]



S[n]

Statistical assumptions

- The input u[n] is a vector WGN sequence i.e

  u[n] is a sequence of uncorrelated jointly Gaussian vectors with E(u[n])=0 and

$$E(u[m]u^T[n])=0 \qquad m \neq n$$

- The initial state s[-1] is a random vector with

$$s[-1] \sim N(\mu_s, \sigma_s^2)$$

Two DC Power Supplies

$$S_1[n] = a_1 S_1[n-1] + u_1[n]$$
$$S_2[n] = a_2 S_2[n-1] + u_2[n]$$

$$\underbrace{\begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}}_{\mathbf{s}[n]} = \underbrace{\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} s_1[n-1] \\ s_2[n-1] \end{bmatrix}}_{\mathbf{s}[n-1]} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} u_1[n] \\ u_2[n] \end{bmatrix}}_{\mathbf{u}[n]}$$

Kalman filter

- Here estimator $\hat{s}[n]$ is computed based on the estimator for previous time sample $\hat{s}[n-1]$ and so recursive in nature.
- Consider the scalar state equation and scalar observation equation

$$s[n] = a\, s[n-1] + u[n]$$

$$x[n] = s[n] + w[n]$$

where $u[n]$ is $N(0,\sigma_u^2)$ $w[n]$ is $N(0,\sigma_n^2)$

we assume $s[-1]$, $u[n]$ and $w[n]$ are all independent.

Finally we assume $s[-1]$ is $N(\mu_s,\sigma_s^2)$

- We have to estimate s[n] based on observations {x[0],x[1],….,x[n]}.
- Estimator of s[n] based on observations {x[0],x[1],…,x[m]} is denoted by s[n|m]
- Criterion of optimality is minimum Bayesian MSE

$$E[(s[n]-\hat{s}[n|n])^2]$$

the expectation is w.r.t p(x[0],x[1],..x[n],s[n]).

- $\hat{s}[n|n]=E[s[n]|x[0],x[1],\ldots,x[n])$.

  let $\mathbf{X}[n]=[x[0]\ x[1]\ \ldots x[n]]^T$

  $$\tilde{x}[n]= x[n]-\hat{x}[n-1]$$

  thus $\hat{s}[n|n]$ can be written as

  $$\hat{s}[n|n]=E[s[n]|\mathbf{X}[n-1],\tilde{x}[n]]$$

- Since $\mathbf{X}[n-1]$ and $\tilde{x}[n]$ are uncorrelated

  $$\hat{s}[n|n]=E[s[n]|\mathbf{X}[n-1]]+E[s[n]|\tilde{x}[n]]$$

- We denote E[s[n]|**X**[n-1]),prediction of s[n],by ŝ[n|n-1]

$$\hat{s}[n|n] = \hat{s}[n|n-1] + E(s[n]|\tilde{x}[n])$$

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \end{aligned}$$

where

$$K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}.$$

- Since x[n]= s[n]+w[n] we have

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

the gain factor k(n) becomes

$$k[n] = \frac{E[(s[n] - \hat{s}[n|n-1])^2]}{\sigma_n^2 + E[(s[n] - \hat{s}[n|n-1])^2]}$$

- Let M[n|n] is the minimum prediction error .
- Then k[n] becomes

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

By solving we get

$M[n|n-1]=a^2 M[n-1|n-1]+\sigma_u^2$

Finally we get

$M[n|n]=(1-k[n])M[n|n-1]$.

Prediction:

$$\hat{s}[n|n-1] = a\hat{s}[n-1|n-1].$$

Minimum Prediction MSE:

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2.$$

Kalman Gain:

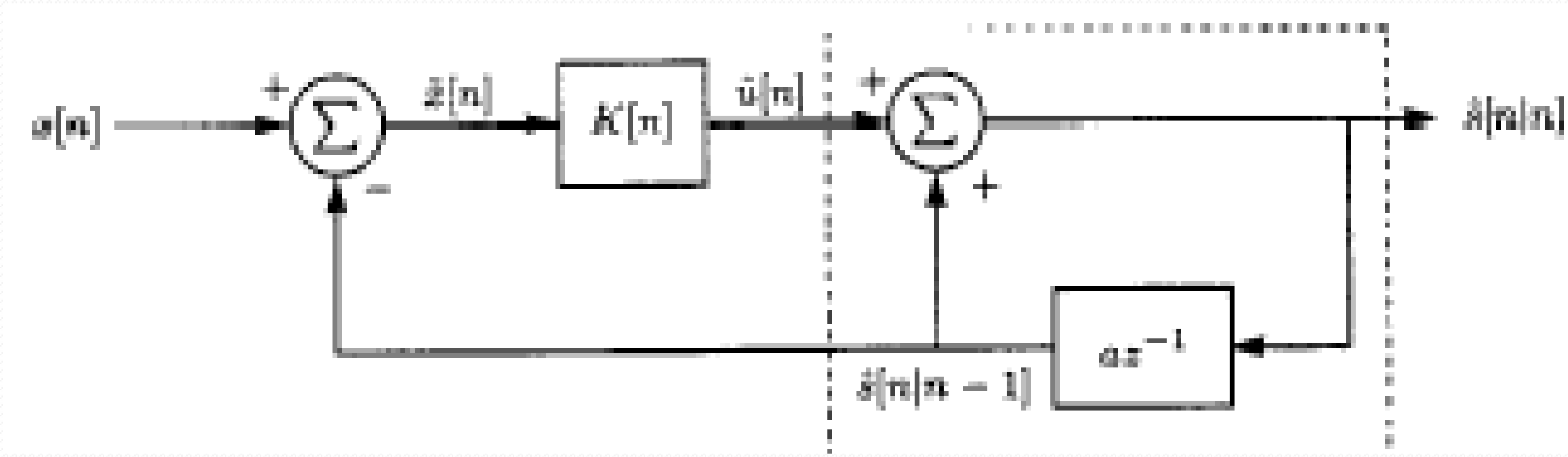
$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}.$$

Correction:

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

Minimum MSE:

$$M[n|n] = (1 - K[n])M[n|n-1].$$



Dynamic model

fig. Scalar state observation Kalman filter and relationship to dynamic model

Properties

- ❖ The kalman filter is a time varying linear filter.
- ❖ The kalman filter provides its own performance measure.
- ❖ The prediction stage increases the error, while the correction stage decreases it.
- ❖ Prediction is an integral part of the Kalman filter.
- ❖ The Kalman filter is driven by uncorrelated innovation sequence

THANK YOU