# A TERM PAPER REPORT
# ON
# KALMAN FILTER

By


***B. Sivaprasad (Y8104059)***

***CH. Venkata Karunya (Y8104066)***

**Department of Electrical Engineering**
**Indian Institute of Technology, Kanpur**
**Kanpur-208016**

# SCALAR KALMAN FILTER

## Introduction:

In practical communication signals are always corrupted due to various reasons but we want less noise signal. So we need to estimate the desired signal by considering a signal which is corrupted by noise. Hence some process should be adopted to reduce noise in the signal. One of the methods can be Statistical Filter designing approach. Filter is used to reduce the amount of noise present in a signal by comparison with an estimation of desired noise less signals.

Consider a basic model, DC Power supply model in which output of supply is around some value but not constant when we are measuring using voltmeter. This unstablisation is because of noise created by Power supply or voltmeter itself.

Consider $x[n] = A + w[n]$

Here A is true value. So here we can get MVU estimator by considering average of x[n] measurements. Hence in this case very easy to get estimator. We get x[n] measurements which are corrupted by noise w[n] , from these we need to estimate A. Even though the power supply should generate a constant voltage of A in practice the voltage will vary due to the effects of temperature, component aging…etc.

Hence $x[n] = A[n] + w[n]$ i.e.

A[n] is true voltage at time 'n'. In this case estimation problem became considerably more complicated since we need to estimate A[n] for n=0, 1…..N-1 instead of just single parameter A. Here we can't take averaging estimator. If we take x[n] as estimator it will be inaccurate due to lack of averaging. Infact the variability will be identical to that of noise. We can use the information that successive samples of A[n] will not be too different, leading us to conclude that they display high degree of correlation. The imposition of a correlation constraint will prevent the estimate of A[n].i.e. present sample depends on the previous sample in this case. Now analyse gauss markov process.

## First order Gauss-Markov Process:

$$S[n]=a\,S[n-1]+u[n] \quad \text{for} \quad n \geq 0$$

i.e. signal at n is depends on value at n-1 and control signal u[n]. we have to assume initial condition S[-1] which is independent of u[n].

Consider $s[-1] \approx N(u_s, \sigma^2)$ and $u[n] \approx N(0, \sigma_u^2)$

The current output s[n] depends only on the state of system at the previous time s[n-1] and it summarizes the effect of all past inputs to the system. As $n \to \infty$ we can consider this process as WSS process. From the equation

$$S[n] = a^{n+1} S[-1] + \sum_{k=0}^{n} a^k u[n-k]$$

We see that S[n] is linear function of S[-1] and the driving noise inputs from the initial time to the present time. From this we get

$$E(s[n]) = a^{n+1} E(S[-1]) = a^{n+1} u_s$$

Covariance between samples:

$$C_s(m, n) = a^{m+n+2} \sigma^2 + \sigma_u^2 \sum_{k=m-n}^{m} a^{2k+n-m}$$

Thus $\text{var}(S[n]) = C_s(n, n) = a^{2n+2} \sigma^2 + \sigma_u^2 \sum_{k=o}^{n} a^{2k}$

However as $n \to \infty$ $\quad E(s[n]) = 0$

$$\text{Var}(S[n]) = \sigma_u^2 a^{m-n} / 1 - a^2$$

we can get mean and variance in another way like

$$E(S[n]) = a\,E(S[n-1])$$

$$\text{var}(S[n]) = a^2 \text{var}(S[n-1]) + \sigma_u^2$$

From these we are able to generate the conclusion that the mean and variance of S[n] only on previous sample S[n-1] and input u[n].

Similarly this discussion can be extended to Pth order gauss markov process.

$$S[n] = \sum_{k=1}^{p} a[k] S[n-k] + u[n]$$

These Gauss Markov process observation results can be effectively use in designing a scalar kalman filter.

## Scalar Kalman filter:

The scalar Gauss-Markov process used in the previous section had the form

$$S[n]=a\,S[n-1]+u[n] \qquad \text{for} \quad n \geq 0$$

The operation of estimating s[n] based on the data {x[0],x[1],….x[n]} is referred to as filtering. In Kalman filter we estimate ŝ[n] based on the estimator for the previous time sample ŝ[n-1], which is a recursive in nature.

Consider the scalar state equation and scalar observation equation

$$s[n] = a\,s[n-1] + u[n]$$

$$x[n] = s[n] + w[n] \qquad \text{------- (1)}$$

Where, u[n] is zero mean Gaussian noise with independent samples and $E[u^2[n]] = \sigma_u^2$, w[n] is zero mean Gaussian noise with independent samples and $E[w^2[n]] = \sigma_w^2$. We further assume that s[-1],u[n] and w[n] are all independent. Finally we assume $s[-1] \approx N(u_s, \sigma_s^2)$. To simplify derivation we assume $\sigma_s = 0$. We determine s[n] based on observations {x[0],x[1],….x[n]}, i.e. to filter x[n] to produce ŝ[n]. We denote the estimator of s[n] based on the observations {x[0],x[1],….x[m]} will be denoted by ŝ[n|m].

We use the following criterion of optimality for minimizing the error

$$E[\{s[n]-\hat{s}[n|n]\}^2]$$

Where the expectation is with respect to p(x[0],x[1],….x[n],s[n])

But the MMSE is just the mean of the posterior PDF or

$$\hat{S}[n|n] = E[s[n]|x[0],x[1],\ldots x[n]]. \qquad \text{-------- (2)}$$

Let $\mathbf{X}[n] = [x[0],x[1],\ldots X[n]]^T$ and $\tilde{x}[n]$ denote the innovation part of x[n] i.e. uncorrelated with the previous samples {x[0],x[1],….x[n-1]} or

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1] \qquad \text{---------(3)}$$

Therefore we can write x[n] as

$$x[n] = \tilde{x}[n] + \hat{x}[n|n-1]$$

$$= \tilde{x}[n] + \sum_{k=0}^{n-1} a_k x[k]$$

Where $a_k$'s are the optimal weighting coefficients of the MMSE estimator of x[n] based on {x[0],x[1],….x[n-1]}. Now we can write eqn (2) as

$$\hat{s}[n|n] = E(s[n]|\boldsymbol{X}[n-1], \tilde{x}[n]) \quad \text{------------(4)}$$

because **X**[n-1] and $\tilde{x}$[n] are uncorrelated we can write the above eqn as

$$\hat{s}[n|n] = E(s[n]|\boldsymbol{X}[n-1]) + E(s[n]|\tilde{x}[n]) \quad \text{------------(5)}$$

we denote the first term on the right hand side as ŝ[n|n-1]

therefore $\hat{s}[n|n-1] = E(s[n]|\mathbf{X}[n-1])$

$= E(a\, s[n-1] + u[n]|\mathbf{X}[n-1])$

$= a\hat{s}[n-1|n-1]$

Since u[n] is independent of X[n-1].

Thus eqn (5) becomes

$$\hat{s}[n|n] = a\hat{s}[n-1|n-1] + E(s[n]|\, \tilde{x}[n]) \quad \text{-----------(6)}$$

since we are assuming Gaussian distributions the MMSE becomes linear and the second term in the above equation becomes

$$E(s[n]|\, \tilde{x}[n]) = \mathbf{K}[n]\, \tilde{x}[n]$$

$$= \mathbf{K}[n](x[n] - \hat{x}[n|n-1]) \quad \text{------------(7)}$$

where $\mathbf{K}[n] = \dfrac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}$

but x[n] = s[n] +w[n]

therefore $\hat{x}[n|n-1] = \hat{s}[n|n-1] + \hat{w}[n|n-1]$

$$= \hat{s}[n|n-1]$$

since $\hat{w}[n|n-1] = 0$ due to w[n] being independent from **X**[n-1]. Thus

$$E(s[n]|\tilde{x}[n] = \boldsymbol{K}[n](x[n] - \hat{s}[n|n-1]) \qquad \text{------------ (8)}$$

From equation (7) we have

$$\hat{s}[n|n] = \hat{s}[n|n-1] + \mathbf{K}[n](x[n]-\hat{s}[n|n-1]) \qquad \text{-------------(9)}$$

from equation (7) the gain factor, **K**[n], is

$$\mathbf{K}[n]=\frac{E[s[n](x[n]-\hat{s}[n|n-1])]}{E[(x[n]-\hat{s}[n|n-1])^2]} \qquad \text{------------(10)}$$

To evaluate this we need two results

1. $E[s[n](x[n]-\hat{s}[n|n-1])] = E[(s[n] - \hat{s}[n|n-1])(x[n]-\hat{s}[n|n-1])]$

2. $E[w[n](s[n]-\hat{s}[n|n-1])] = 0$.

From the above two results we get

$$\mathbf{K}[n]= \frac{E[(s[n]-\hat{s}[n|n-1])(x[n]-\hat{s}[n|n-1])]}{E[(s[n]-\hat{s}[n|n-1]+w[n])^2]} \qquad \text{------------(11)}$$

But the numerator is just the minimum MSE incurred when s[n] is estimated based on previous data. We denote this by M[n|n-1].

$$\boldsymbol{K}[n] = \frac{M[n|n-1]}{{\sigma_n}^2+M[n|n-1]} \qquad \text{----------(12)}$$

Using the equation (6) and noting that

$$E[(s[n-1]-\hat{s}[n-1|n-1]u[n]] = 0$$

We get $\quad M[n|n-1] = a^2M[n-1|n-1] + \sigma_u^2 \qquad \text{------------(13)}$

Finally we require a recursion for M[n|n] . using equation (9) we have

$$M[n|n]= (1-\mathbf{K}[n])M[n-1|n-1] \qquad \text{-------------(14)}$$

This completes the derivation of scalar state scalar observation Kalman filter.

**Prediction:**

$$\hat{s}[n|n-1] = a\hat{s}[n-1|n-1].$$

**Minimum Prediction MSE:**

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2.$$

**Kalman Gain:**

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}.$$

**Correction:**

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1]).$$

**Minimum MSE:**

$$M[n|n] = (1 - K[n])M[n|n-1].$$

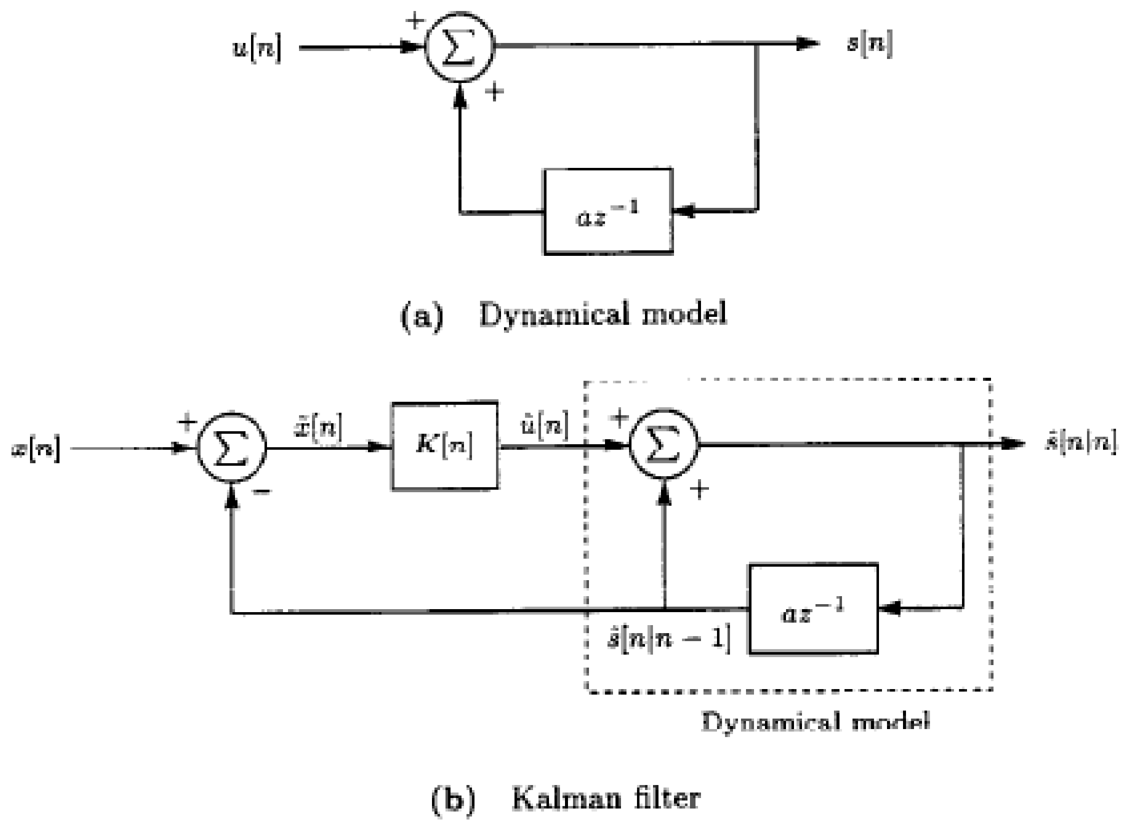


Fig. Scalar state scalar observation Kalman filter and relationship to dynamic model.

## Properties of Kalman filter:

1. The kalman filter is time varying filter as K[n] depends on n.
2. The kalman filter provides its own performance measure. This is because M[n|n] depends only on M[n|n-1] and K[n], which are independent of data.
3. The prediction stage increases the error, while the correction stage decreases it.
4. Prediction is an integral part of Kalman filter.
5. The Kalman filter is driven by the uncorrelated innovation sequence