A TERM PAPER REPORT
ON
WIENER FILTERING

By

N.Srikanth(Y8104060)
M.Manikanta PhaniKumar(Y8104031)

Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Kanpur-208016

INTRODUCTION

Noise is present in many situations of daily life for ex: Microphones will record noise and speech. But our goal is to reconstruct original signal. One of the methods, called “WINER FILTERING” is explained here.

Wiener filtering is a method to estimate the original signal as close as possible from the signals degraded by additive white noise. It is one which based on the linear minimum mean square error (LMMSE) estimator. Calculation of the Wiener filter requires the assumption that the signal and noise processes are second-order stationary. For this description, the data is WSS with zero mean will be considered.

Here we study mainly three problems and they are listed below.

1) Smoothing

2) Filtering

3) Prediction

To solve all three above problems we use, LMMSE estimator for x[n] based on the data set { x[0],x[1],…….x[N-1] } , where x[n]= s[n] + w[n] with E(x)=E(Θ).

$$\hat{\Theta} = C_{\Theta x} \; C_{xx}^{-1} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and the minimum MSE matrix given by

$$M_{\hat{\theta}} = C_{\theta\theta} - C_{\Theta x} \; C_{xx}^{-1} \, C_{x\theta} \;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

1. SMOOTHING:

In smoothing Θ=s[n] is to be estimated for n=0,1,………N-1, based on the entire data set { x[0],x[1],…….x[N-1] } , where x[n]= s[n] + w[n]. In this estimation cannot be obtained until all the data has been collected. Here we make an assumption that signal and noise are uncorrelated.
Thus we have

$$\mathbf{C}_{xx} = \mathbf{R}_{xx} = \mathbf{R}_{ss} + \mathbf{R}_{ww}.$$

$$\mathbf{C}_{\theta x} = E(\mathbf{s}\mathbf{x}^T) = E(\mathbf{s}(\mathbf{s}+\mathbf{w})^T) = \mathbf{R}_{ss}.$$

So, substituting above equation in (1), then the wiener estimation of the signal is

$$\hat{\mathbf{s}} = \mathbf{R}_{ss}(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}\mathbf{x}.$$

The NxN matrix

$$\mathbf{W} = \mathbf{R}_{ss}(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}$$

is referred to as wiener smoothing matrix.

and the corresponding minimum MSE matrix is

$$\begin{aligned} \mathbf{M}_{\hat{s}} &= \mathbf{R}_{ss} - \mathbf{R}_{ss}(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}\mathbf{R}_{ss} \\ &= (\mathbf{I} - \mathbf{W})\mathbf{R}_{ss}. \end{aligned}$$

For example if N=1, we would estimate S[0] based on X[0]=S[0]+w[0]. Then, the wiener smoother W is given by

$$\begin{aligned} W &= r_{ss}[0] \,/\, (r_{ss}[0] + r_{ww}[0]) \\ &= \eta / (\eta+1) \end{aligned}$$

Where $\eta = r_{ss}[0] / r_{ww}[0]$ is the SNR

For high SNR so what W→1, we have Ŝ[0]→X[0], while for a low SNR so what W→0, we have Ŝ[n]→0.

The corresponding minimum MSE is

$$\begin{aligned} M_{\hat{s}} &= (1 - W)\, r_{ss}[0] \\ &= (1 - \eta/(\eta+1))\, r_{ss}[0] \end{aligned}$$

Which for these two extremes is either 0 for a high SNR, $r_{xx}[0]$ for a low SNR.

2. FILTERING:

In filtering the signal sample is estimated based on the present and past data only. We estimate $\theta = s[n]$ based on $x[n] = s[m]+w[m]$ for $m=0,1,2\ldots\ldots n$ i.e. using the data $\{x[0],x[1],\ldots\ldots x[n]\}$.As before, $x[n] = s[n] + w[n]$,where $s[n]$ and $w[n]$ are signal and noise processes that are uncorrelated with each other. Thus,

$$\mathbf{C}_{xx} = \mathbf{R}_{ss} + \mathbf{R}_{ww}$$

where R_{ss} & R_{ww}, are (n + 1) x (n + 1) autocorrelation matrices.

$$\begin{aligned}\mathbf{C}_{\theta x} &= E\left(s[n]\left[\begin{array}{cccc} x[0] & x[1] & \ldots & x[n]\end{array}\right]\right)\\ &= E\left(s[n]\left[\begin{array}{cccc} s[0] & s[1] & \ldots & s[n]\end{array}\right]\right)\\ &= [r_{ss}[n]\, r_{ss}[n-1] \ldots r_{ss}[0]].\end{aligned}$$

Substituting above equations in (1) we get the estimator of the signal as follows

$$\hat{S}[n] = r'^{T}_{ss} (R_{ss} + R_{ww})^{-1} x$$

$$\hat{S}[n] = a^T x$$

where a is (n + 1) x 1 vector of weights is seen to be

$$a = (R_{ss} + R_{ww})^{-1} r'_{ss}$$

To make the filtering correspondence we let $h^{(n)}[k] = a_{n-k}$. then

$$\begin{aligned}\hat{s}[n] &= \sum_{k=0}^{n} a_k x[k]\\ &= \sum_{k=0}^{n} h^{(n)}[n-k]x[k]\end{aligned}$$

$$\hat{s}[n] = \sum_{k=0}^{n} h^{(n)}[k]x[n-k]$$

where $h^{(n)}[k]$ is the time varying FIR filter. To explicitly find the impulse response **h** we note that since

$$(\mathbf{R}_{ss} + \mathbf{R}_{ww})\,\mathbf{a} = \mathbf{r}'_{ss}$$

$$(\mathbf{R}_{ss} + \mathbf{R}_{ww})\,\mathbf{h} = \mathbf{r}_{ss}$$

where r_{ss} = [r_{ss}[O], r_{ss} [1],……., r_{ss} [n]]T and above equation can be written as set of linear equations as folloes.

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[n] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[n-1] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[n] & r_{xx}[n-1] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h^{(n)}[0] \\ h^{(n)}[1] \\ \vdots \\ h^{(n)}[n] \end{bmatrix} = \begin{bmatrix} r_{ss}[0] \\ r_{ss}[1] \\ \vdots \\ r_{ss}[n] \end{bmatrix}$$

These are the wiener-Hopf filtering equations.

For large enough n it can be shown that the filter becomes time invariant, the Wiener-Hopf filtering equations can be written as

$$\sum_{k=0}^{\infty} h[k] r_{xx}[l-k] = r_{ss}[l] \qquad l = 0, 1, \ldots .$$

where we have used the property $r_{xx}[-k] = r_{xx}[k]$**.** As n tends to ∞, we have upon replacing the time varying impulse response $\mathbf{h^{(n)}[k]}$ by its time invariant version **h[k]**

$$\sum_{k=0}^{\infty} h[k] r_{xx}[l-k] = r_{ss}[l] \qquad l = 0, 1, \ldots .$$

The same set of equations result if we attempt to estimate s[n] based on the present and infinite past. This is termed the infinite wiener filter.

Let

$$\hat{s}[n] = \sum_{k=0}^{\infty} h[k] x[n-k]$$

and use the orthogonality principle.

Then, E[(S[n] - Ŝ[n]) x[n-l])] =0 ; l=0,1…….

Hence,

$$E\left(\sum_{k=0}^{\infty} h[k] x[n-k] x[n-l]\right) = E(s[n] x[n-l])$$

and therefore, the equations to be solved for the infinite wiener filter impulse response are

$$\sum_{k=0}^{\infty} h[k] r_{xx}[l-k] = r_{ss}[l] \qquad l = 0, 1, \ldots .$$

The problem of estimating s[n] based on x[m] for $0 \leq m \leq n$ as n tends to ∞ is really just that of using the present and infinite past to estimate the current sample. The time invariance of the filter makes the solution independent of which sample is to be estimated or independent of n**.** The above equation can be solved by using Fourier transform techniques if h[k] is casual.

Now if we again consider smoothing problem, we get the same sot of equations in which s[n] is to be estimated based on {. . . , x [- 1] , x[O] , x [1] , … } or x[k] for all k.

In this case the smoothing estimator takes the form

$$\hat{S}[n] = \sum_{k=-\infty}^{\infty} a_k \, x[k]$$

And by letting $h[k] = a_{n-k}$ we have the convolution sum

$$\hat{S}[n] = \sum_{k=-\infty}^{\infty} h[k] \, x[n-k]$$

The wiener equation becomes

$$\sum_{k=-\infty}^{\infty} h[k] \, r_{xx}[l-k] = r_{ss}[l] \quad ; \; -\infty < l < \infty$$

The difference from the filtering case is that now the equations must be satisfied for all, and there is no constraint that h[k] must be causal. Hence we can use Fourier transform techniques to solve for the impulse response, then

$$H(f) = P_{ss}(f) \, / \, P_{xx}(f)$$

$$= P_{ss}(f) \, / \, (P_{ss}(f) + P_{ww}(f))$$

If we define SNR as $\eta(f) = P_{ss}(f) / P_{ww}(f)$

Then the optimal filter frequency response becomes

$$H(f) = \eta(f) \, / \, (\eta(f)+1)$$

Clearly, the filter response satisfies $0 < H(f) < 1$, and the wiener smoother response is $H(f) \approx 0$ when $\eta(f) \approx 0$ and $H(f) \approx 1$ when $n(f) \rightarrow \infty$.

3. PREDICTION:

In which we estimate $\Theta = x[n-1+l]$ for $l > 1$ based on x=([x[0],x[1]………… …..x[N-1]). The resulting estimator is termed the l-step linear predictor.

Then

$$\hat{x}[N-1+l] = \mathbf{r}'^{T}_{xx}\mathbf{R}^{-1}_{xx}\mathbf{x}.$$

$$\mathbf{a} = \mathbf{R}^{-1}_{xx}\mathbf{r}'_{xx}$$

$$\hat{x}[N-1+l] = \sum_{k=0}^{N-1} a_k x[k].$$

let $h[N-k]=a_k$ to allow filtering interpretation

$$\begin{aligned} \hat{x}[N-1+l] &= \sum_{k=0}^{N-1} h[N-k]x[k] \\ &= \sum_{k=1}^{N} h[k]x[N-k] \end{aligned}$$

The minimum MSE for l-step linear predictor is

$$M_{\hat{x}} = r_{xx}[0] - \mathbf{r}'^{T}_{xx}\mathbf{R}^{-1}_{xx}\mathbf{r}'_{xx}$$

$$\begin{aligned} M_{\hat{x}} &= r_{xx}[0] - \mathbf{r}'^{T}_{xx}\mathbf{a} \\ &= r_{xx}[0] - \sum_{k=0}^{N-1} a_k r_{xx}[N-1+l-k] \\ &= r_{xx}[0] - \sum_{k=0}^{N-1} h[N-k] r_{xx}[N-1+l-k] \\ &= r_{xx}[0] - \sum_{k=1}^{N} h[k] r_{xx}[k+(l-1)]. \end{aligned}$$

Assume that x[n] is an AR(1) process with ACF, then

$$r_{xx}[k] = \frac{\sigma_u^2}{1-a^2[1]}(-a[1])^{|k|}$$

The impulse response is

$$h[k] = \begin{cases} (-a[1])^l & k=1 \\ 0 & k=2,3,\ldots,N \end{cases}$$

Therefore the L step predictor is

$$\hat{x}[(N-1)+l] = (-a[1])^l\, x[N-1].$$

and the minimum MSE for L step predictor is

$$\begin{aligned} M_{\hat{x}} &= r_{xx}[0] - h[1] r_{xx}[l] \\ &= \frac{\sigma_u^2}{1-a^2[1]} - (-a[1])^l \frac{\sigma_u^2}{1-a^2[1]} (-a[1])^l \\ &= \frac{\sigma_u^2}{1-a^2[1]} \left(1 - a^{2l}[1]\right). \end{aligned}$$

The predictor decays to zero with increase in L, since $|a[1]|<1$ (for system is stable). This is also reflected in the minimum MSE, which is smallest for L=1 and increases for larger one.