Project Report

Statistical Signal Processing

The Generalized Correlation Method for Estimation of Time Delay

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Table of Contents

1. Fundamentals

2. Processor Interpretation

3. Roth Processor

4. Smoothed Coherence Transform(SCOT)

5. The Phase Transform(PHAT)

6. Eckart Filter

7. Interpretation of Low SNR of ML Estimator

8. Conclusion
Fundamentals

Signal emanating from a remote source and monitored in the presence of noise at two spatially separated sensors can be mathematically modeled as

\[ x_1(t) = s_1(t) + n_1(t) \]
\[ x_2(t) = \alpha s_1(t + D) + n_2(t), \]

where \( s_1(t) \), \( n_1(t) \), and \( n_2(t) \) are real, jointly stationary random processes. Signal \( s_1(t) \) is assumed to be uncorrelated with noise \( n_1(t) \) and \( n_2(t) \).

There are many applications in which it is of interest to estimate the delay \( D \). This paper proposes a maximum likelihood (ML) estimator and compares it with other similar techniques. While the model of the physical phenomena’ presumes stationary, the techniques to be developed herein are usually employed in slowly varying environments where the characteristics of the signal and noise remain stationary only for finite observation time \( T \).

Further, the delay \( D \) and attenuation \( \alpha \) may also change slowly. The estimator is, therefore, constrained to operate on observations of a finite duration. Another important consideration in estimator design is the available amount of a priori knowledge of the signal and noise statistics. In many problems, this information is negligible. For example, in passive detection, unlike the usual communications problems, the source spectrum is unknown or only known approximately.
One common method of determining the time delay $D$ and, hence, the arrival angle relative to the sensor axis [l] is to compute the cross correlation function given by

$$R_{x_1, x_2}(\tau) = E[x_1(t) x_2(t - \tau)],$$

Because of finite duration only, above can only be estimated. Considering ergodic process, it is written as

$$\hat{R}_{x_1, x_2}(\tau) = \frac{1}{T - \tau} \int_{\tau}^{T} x_1(t) x_2(t - \tau) \, dt,$$

Where $T$ represents the observation interval. In order to improve the accuracy of the delay estimate $D$, it is desirable toprefilter $x_1(t)$ and $x_2(t)$ prior to the integration. As shown in the below figure, $x_i$ may be filtered through $H_i$ to yield $y_i$ for $i = 1, 2$. The resultant $y_i$ are multiplied, integrated, and squared for a range of time shifts, $T$, until the peak is obtained. The time shift causing the peak is an estimate of the true delay $D$.

This paper provides for a generalized correlation through the introduction of the filters $H_1(f)$ and $H_2(f)$ which, when properly selected, facilitate the estimation of delay. The time shift causing the peak is an estimate of the true delay $D$. 
The cross power spectral density function by the well-known Fourier transform relationship is given by

\[ R_{x_1 x_2}(\tau) = \int_{-\infty}^{\infty} G_{x_1 x_2}(f) e^{j2\pi f \tau} \, df. \]

Generalized correlation between \( x_1(t) \) and \( x_2(t) \) is given by

\[ R_{y_1 y_2}^{(g)}(\tau) = \int_{-\infty}^{\infty} \psi_g(f) G_{x_1 x_2}(f) e^{j2\pi f \tau} \, df, \]

where

\[ \psi_g(f) = H_1(f) H_2^*(f) \]

In practice, only an estimate of \( G_{x_1 x_2}(f) \) can be obtained from finite observations of \( x_1(t) \) and \( x_2(t) \). Consequently, the integral

\[ \hat{R}_{y_1 y_2}^{(g)}(\tau) = \int_{-\infty}^{\infty} \psi_g(f) \hat{G}_{x_1 x_2}(f) e^{j2\pi f \tau} \, df \]
**Processor Interpretation**

Cross Correlation of $x_1$ and $x_2$

$$R_{x_1, x_2} (\tau) = \alpha R_{s_1, s_1} (\tau - D) + R_{n_1, n_2} (\tau).$$

Fourier Transform of above gives

$$G_{x_1, x_2} (f) = \alpha G_{s_1, s_1} (f) e^{-j2\pi fD} + G_{n_1, n_2} (f).$$

Interpretation:

$$R_{x_1, x_2} (\tau) = \alpha R_{s_1, s_1} (\tau) \otimes \delta(t - D),$$

One interpretation is that the delta function has been spread by the Fourier transform of signal spectrum.

For a single delay this may not be a serious problem. However, when the signal has multiple delays, the true cross correlation is given by

$$R_{x_1, x_2} (\tau) = R_{s_1, s_1} (\tau) \otimes \sum_i \alpha_i \delta(\tau - D_i).$$
1. Roth Processor

The weighting proposed by Roth

\[ \psi_R(f) = \frac{1}{G_{x_1x_1}(f)} \]

Cross Correlation for the above weighting function is given by

\[ \hat{R}_{y_1y_2}^{(R)}(\tau) = \int_{-\infty}^{\infty} \frac{\hat{G}_{x_1x_1}(f)}{G_{x_1x_1}(f)} e^{i2\pi f \tau} df. \]

Above relation can be written as below.

\[ R_{y_1y_2}^{(R)}(\tau) = \delta(\tau - D) \otimes \int_{-\infty}^{\infty} \frac{\alpha G_{s_1s_1}(f)}{\{G_{s_1s_1}(f) + G_{n_1n_1}(f)\}} \cdot e^{i2\pi f \tau} df. \]

Effect of above Weighting Function:

Desirable effect of suppressing those frequency regions where \( G_{n_1n_1}(f) \) is large.

2. Smoothed Coherence Transform (SCOT)

For SCOT, weighting function

\[ \psi_s(f) = 1/\sqrt{G_{x_1x_1}(f)G_{x_2x_2}(f)}. \]
This weighting gives the SCOT

\[ \hat{R}_{y_1y_2}^{(s)}(\tau) = \int_{-\infty}^{\infty} \hat{\gamma}_{x_1x_2}(f) e^{j2\pi f\tau} df, \]

where

\[ \hat{\gamma}_{x_1x_2}(f) \equiv \frac{\hat{G}_{x_1x_2}(f)}{\sqrt{\hat{G}_{x_1x_1}(f) \hat{G}_{x_2x_2}(f)}}. \]

When \( G_{x_1x_1}(f) = G_{x_2x_2}(f) \), the SCOT is equivalent to the Roth processor.

3. The Phase Transform (PHAT)

To avoid the spreading, PHAT uses weighting

\[ \psi_p(f) = \frac{1}{|G_{x_1x_2}(f)|} \]

With uncorrelated noise \( \text{i.e., } \hat{G}_{n_i,n_i}(f) = 0 \). Cross Correlation is given by

\[ R_{y_1y_2}^{(p)}(\tau) = \delta(t - D). \]

since

\[ \frac{\hat{G}_{x_1x_2}(f)}{|G_{x_1x_2}(f)|} = e^{j\theta(f)} = e^{j2\pi fD} \]

For the model as in fig1 with uncorrelated noises, PHAT does not suffer the spreading as other processors.

Estimate of \( R_{y_1y_2}^{(p)}(\tau) \) will not be a delta function if

\[ \hat{G}_{x_1x_2}(f) \neq \hat{G}_{x_1x_2}(f), \quad \theta(f) \neq 2\pi fD \]

in some frequency band, then \( \theta(f) \) is undefined in that band and the estimate of the phase is uniform \([ -\pi, \pi ]\) rad.
So need for additional weight to compensate for the presence or absence of the signal. SCOT assigns weight according to signal and noise characteristics.

4. Eckart Filter

Weighting is given by

$$
\psi_E(f) = \frac{\alpha G_{s_1} (f)}{G_{n_1 n_1} (f) G_{n_2 n_2} (f)},
$$

This possesses some of the qualities of SCOT such as suppressing frequency bands of high noise. This filter attaches zero weigh to bands where $G_{s_1 s_1} (f) = 0$ unlike PHAT.

5. The HT Processor (Hannon and Thomson)

ML estimator selects as the estimate of delay the value of $\tau$ at which

$$
R^{(HT)}_{y_1, y_2} (\tau) = \int_{-\infty}^{\infty} \hat{G}_{x_1 x_2} (f) \frac{1}{|G_{x_1 x_2} (f)|} \cdot \frac{|\gamma_{12} (f)|^2}{[1 - |\gamma_{12} (f)|^2]} \cdot e^{j2\pi f \tau} \, df
$$

achieves peak.

Weight is given by

$$
\psi_{HT} (f) = \frac{1}{|G_{x_1 x_2} (f)|} \cdot \frac{|\gamma_{12} (f)|^2}{[1 - |\gamma_{12} (f)|^2]},
$$
Interpretation of Low SNR of ML Estimator

Good delay estimation is most difficult in the case of low SNR.

For Low SNR

\[
\frac{G_{s_1 s_1}(f)}{G_{n_1 n_1}(f)} \ll 1 \quad \text{and} \quad \frac{G_{s_1 s_1}(f)}{G_{n_2 n_2}(f)} \ll 1,
\]

It follows that

\[
\psi_{HT}(f) \approx \frac{G_{s_1 s_1}(f)}{G_{n_1 n_1}(f) G_{n_2 n_2}(f)} = \psi_E(f).
\]

If

\[
G_{n_1 n_1}(f) = G_{n_2 n_2}(f) = G_{nn}(f),
\]

then

\[
\psi_{HT}(f) \approx \frac{G_{s_1 s_1}(f)}{G_{nn}(f)} \left[\psi_s(f) = \left[\frac{G_{s_1 s_1}(f)}{G_{nn}(f)}\right]^2 \psi_p(f).\right.
\]

Thus, under low S/N ratio approximations with alpha = 1, both the Eckart and HT prefilters can be interpreted either as SCOT pre whitening filters with additional S/N ratio weighting or PHAT pre whitening filters with additional S/N ratio squared weighting.
Conclusion:

- HT processor has been shown to be an estimator time delay under usual conditions.
- Under Low SNR, the HT processor is equivalent to Eckart prefiling and cross correlation.
- If the coherence is slowly changing as a function of time, the ML estimation is a cross correlate preceded by prefilters that vary with time.