Project Report

Statistical Signal Processing

The Generalized Correlation Method for Estimation of Time Delay

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Fundamentals

Signal emanating from a remote source and monitored in the presence of noise at two spatially separated sensors can be mathematically modeled as

$$x_1(t) = s_1(t) + n_1(t)$$

$$x_2(t) = \alpha s_1(t + D) + n_2(t),$$

where $s_1(t)$, $n_1(t)$, and $n_2(t)$ are real, jointly stationary random processes. Signal $s_1(t)$ is assumed to be uncorrelated with noise $n_1(t)$ and $n_2(t)$.

There are many applications in which it is of interest to estimate the delay D. This paper proposes a maximum likelihood (ML) estimator and compares it with other similar techniques. While the model of the physical phenomena' presumes stationary, the techniques to be developed herein are usually employed in slowly varying environments where the characteristics of the signal and noise remain stationary only for finite observation time T.

Further, the delay D and attenuation a may also change slowly. The estimator is, therefore, constrained to operate on observations of a finite duration. Another important consideration in estimator design is the available amount of a priori knowledge of the signal and noise statistics. In many problems, this information is negligible. For example, in passive detection, unlike the usual communications problems, the source spectrum is unknown or only known approximately.

One common method of determining the time delay D and, hence, the arrival angle relative to the sensor axis [l] is to compute the cross correlation function given by

$$R_{x_1 x_2}(\tau) = E[x_1(t)\, x_2(t - \tau)],$$

Because of finite duration only, above can only be estimated. Considering ergodic process, it is written as

$$\hat{R}_{x_1 x_2}(\tau) = \frac{1}{T - \tau} \int_{\tau}^{T} x_1(t)\, x_2(t - \tau)\, dt,$$

Where T represents the observation interval. In order to improve the accuracy of the delay estimate D, it is desirable to prefilter xl(t) and x2(t) prior to the integration. As shown in the below figure, xi may be filtered through Hi to yield yi for i = 1, 2. The resultant yi are multiplied, integrated, and squared for a range of time shifts, T, until the peak is obtained. The time shift causing the peak is an estimate of the true delay D.

This paper provides for a generalized correlation through the introduction of the filters H1(f) and H2(f) which, when properly selected, facilitate the estimation of delay. The time shift causing the peak is an estimate of the true delay D.

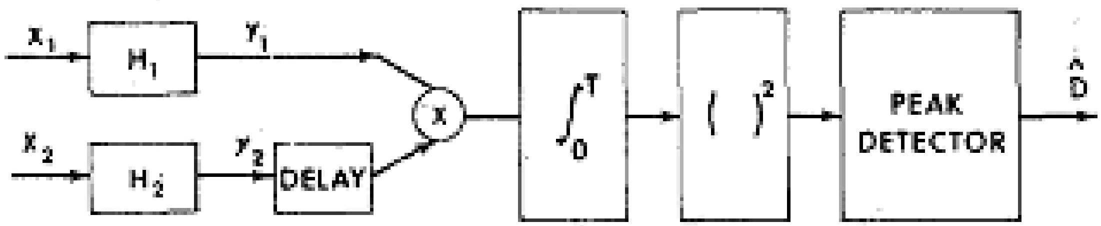

Fig. 1. Received waveforms filtered, delayed, multiplied, and integrated for a variety of delays until peak output is obtained.

The cross power spectral density function by the well-known Fourier transform relationship is given by

$$R_{x_1x_2}(\tau) = \int_{-\infty}^{\infty} G_{x_1x_2}(f)\, e^{j2\pi f\tau}\, df.$$

Generalized correlation between x1(t) and x2(t) is given by

$$R^{(g)}_{y_1y_2}(\tau) = \int_{-\infty}^{\infty} \psi_g(f)\, G_{x_1x_2}(f)\, e^{j2\pi f\tau}\, df,$$

where

$$\psi_g(f) = H_1(f)\, H_2^*(f)$$

In practice, only an estimate of $G_{x1x2}(f)$ can be obtained from finite observations of xl(t) and x2(t). Consequently, the integral

$$\hat{R}^{(g)}_{y_1y_2}(\tau) = \int_{-\infty}^{\infty} \psi_g(f)\, \hat{G}_{x_1x_2}(f)\, e^{j2\pi f\tau}\, df$$

Processor Interpretation

Cross Correlation of x1 and x2

$$R_{x_1 x_2}(\tau) = \alpha R_{s_1 s_1}(\tau - D) + R_{n_1 n_2}(\tau).$$

Fourier Transform of above gives

$$G_{x_1 x_2}(f) = \alpha G_{s_1 s_1}(f)\, e^{-j2\pi f D} + G_{n_1 n_2}(f).$$

Interpretation:

$$R_{x_1 x_2}(\tau) = \alpha R_{s_1 s_1}(\tau) \circledast \delta(t - D),$$

One interpretation is that the delta function has been spread by the Fourier transform of signal spectrum.

For a single delay this may not be a serious problem. However, when the signal has multiple delays, the true cross correlation is given by

$$R_{x_1 x_2}(\tau) = R_{s_1 s_1}(\tau) \circledast \sum_i \alpha_i \delta(\tau - D_i).$$

1. Roth Processor

The weighting proposed by Roth

$$\psi_R(f) = \frac{1}{G_{x_1 x_1}(f)}$$

Cross Correlation for the above weighting function is given by

$$\hat{R}^{(R)}_{y_1 y_2}(\tau) = \int_{-\infty}^{\infty} \frac{\hat{G}_{x_1 x_2}(f)}{G_{x_1 x_1}(f)} e^{j2\pi f\tau}\, df.$$

Above relation can be written as below.

$$R^{(R)}_{y_1 y_2}(\tau) = \delta(\tau - D) \circledast \int_{-\infty}^{\infty} \frac{\alpha G_{s_1 s_1}(f)}{\{G_{s_1 s_1}(f) + G_{n_1 n_1}(f)\}} \cdot e^{j2\pi f\tau}\, df.$$

Effect of above Weighting Function:

Desirable effect of suppressing those frequency regions where $G_{n_1 n_1}(f)$ is large.

2. Smoothed Coherence Transform (SCOT)

For SCOT, weighting function

$$\psi_s(f) = 1/\sqrt{G_{x_1 x_1}(f) G_{x_2 x_2}(f)}.$$

This weighting gives the SCOT

$$\hat{R}^{(s)}_{y_1 y_2}(\tau) = \int_{-\infty}^{\infty} \hat{\gamma}_{x_1 x_2}(f)\, e^{j2\pi f\tau}\, df,$$ where

$$\hat{\gamma}_{x_1 x_2}(f) \triangleq \frac{\hat{G}_{x_1 x_2}(f)}{\sqrt{G_{x_1 x_1}(f)\, G_{x_2 x_2}(f)}}.$$

When $G_{x_1 x_1}(f) = G_{x_2 x_2}(f)$, the SCOT is equivalent to the Roth processor.

3. The Phase Transform (PHAT)

To avoid the spreading, PHAT uses weighting

$$\psi_p(f) = \frac{1}{|G_{x_1 x_2}(f)|}$$

With uncorrelated noise $\text{i.e.}, G_{n_1 n_2}(f) = 0$. Cross Correlation is given by $R^{(p)}_{y_1 y_2}(\tau) = \delta(t - D).$ since $\frac{\hat{G}_{x_1 x_2}(f)}{|G_{x_1 x_2}(f)|} = e^{j\theta(f)} = e^{j2\pi f D}$

For the model as in fig1 with uncorrelated noises, PHAT does not suffer the spreading as other processors.

Estimate of $R^{(p)}_{y_1 y_2}(\tau)$ will not be a delta function if

$\hat{G}_{x_1 x_2}(f) \neq \bar{G}_{x_1 x_2}(f),\ \ \theta(f) \neq 2\pi f\bar{D}$ in some frequency band, then $\theta(f)$ is undefined in that band and the estimate of the phase is uniform $[-\pi, \pi]$ rad.

So need for additional weight to compensate for the presence or absence of the signal. SCOT assigns weight according to signal and noise characteristics.

4. Eckart Filter

Weighting is given by

$$\psi_E(f) = \frac{\alpha G_{s_1 s_1}(f)}{G_{n_1 n_1}(f)\, G_{n_2 n_2}(f)} .$$

This possesses some of the qualities of SCOT such as suppressing frequency bands of high noise. This filter attaches zero weigh to bands where $G_{s_1 s_1}(f) = 0$ unlike PHAT.

5. The HT Processor (Hannon and Thomson)

ML estimator selects as the estimate of delay the value of τ at which

$$R_{y_1 y_2}^{(\mathrm{HT})}(\tau) = \int_{-\infty}^{\infty} \hat{G}_{x_1 x_2}(f) \frac{1}{|G_{x_1 x_2}(f)|} \cdot \frac{|\gamma_{12}(f)|^2}{[1 - |\gamma_{12}(f)|^2]} e^{j2\pi f\tau}\, df$$

achieves peak.

Weight is given by

$$\psi_{\mathrm{HT}}(f) = \frac{1}{|G_{x_1 x_2}(f)|} \cdot \frac{|\gamma_{12}(f)|^2}{[1 - |\gamma_{12}(f)|^2]},$$

Interpretation of Low SNR of ML Estimator

Good delay estimation is most difficult in the case of low SNR.

For Low SNR

$$\frac{G_{s_1 s_1}(f)}{G_{n_1 n_1}(f)} << 1 \quad \text{and} \quad \frac{G_{s_1 s_1}(f)}{G_{n_2 n_2}(f)} << 1,$$

It follows that

$$\psi_{\mathrm{HT}}(f) \cong \frac{G_{s_1 s_1}(f)}{G_{n_1 n_1}(f)\, G_{n_2 n_2}(f)} = \psi_E(f).$$

If $G_{n_1 n_1}(f) = G_{n_2 n_2}(f) = G_{nn}(f),$ then

$$\psi_{\mathrm{HT}}(f) \cong \frac{G_{s_1 s_1}(f)}{G_{nn}(f)}\, [\psi_s(f)] = \left[\frac{G_{s_1 s_1}(f)}{G_{nn}(f)}\right]^2 \psi_p(f).$$

Thus, under low S/N ratio approximations with alpha = 1, both the Eckart and HT prefilters can be interpreted either as SCOT pre whitening filters with additional S/N ratio weighting or PHAT pre whitening filters with additional S/N ratio squared weighting.

Conclusion:

- HT processor has been shown to be an estimator time delay under usual conditions.
- Under Low SNR, the HT processor is equivalent to Eckart prefiltering and cross correlation.
- If the coherence is slowly changing as a function of time, the ML estimation is a cross correlate preceded by prefilters that vary with time.