DOA Estimation Using MUSIC and Root MUSIC Methods

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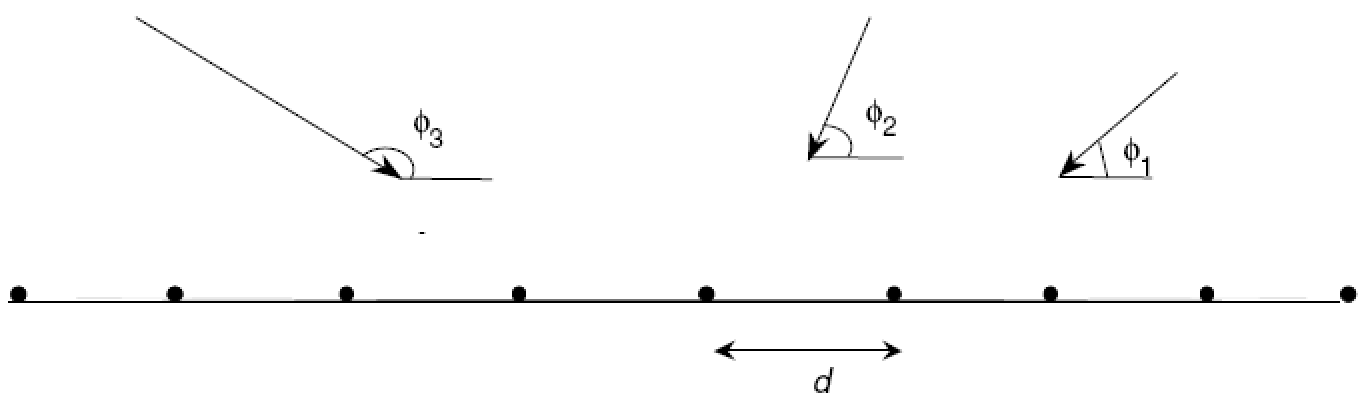
Overview

- Introduction to the DOA problem
- CRB for DOA Estimation
- DOA Estimation using MUSIC
- Root – MUSIC: Model Based Parameter Estimation

Direction of Arrival Estimation

The problem at hand:
Estimate the direction of a signal from the received signals



The DOA estimation problem.

Direction of Arrival Estimation

Problem description:

- A set of incoming signals
- Incoming signal direction, ϕ_i
- A linear, equispaced antenna array with N elements

Direction of Arrival Estimation

Difficulties faced

- No of incoming signals unknown
- Unknown direction and amplitude
- The signals are corrupted by noise

Assumption:

- Number of unknown signals is M
- $M < N$
- White Gaussian Noise

CRB for DOA Estimation

$$\mathbf{x} = \mathbf{v}(\boldsymbol{\theta}) + \mathbf{n}$$

x is length-N vector of received signals

$\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_P]^T$ is the set of parameters

v is a *known* function of parameters

we know that, $\text{var}(\theta_p) \geq \mathbf{J}_{pp}^{-1}$

where

$$\mathbf{J}_{ij} = \mathrm{E}\left\{ \frac{\partial^2}{\partial\theta_i \partial\theta_j} \left[\ln f_{\mathbf{X}}(\mathbf{x}/\boldsymbol{\theta})\right] \right\}$$

J is the Fischer Information Matrix

CRB for DOA Estimation

- Assuming a single signal corrupted by noise,

$$\mathbf{x} = \alpha \mathbf{s}(\phi) + \mathbf{n}$$

where **s** is the steering vector of the signal

n is zero mean Gaussian with covariance $\sigma^2 \mathbf{I}$

Assuming $\alpha = a e^{jb}$

$$\boldsymbol{\theta} = [a,\, b,\, \phi]^T$$

In our case

$$\begin{aligned} \mathbf{v}(\boldsymbol{\theta}) &= \alpha\, \mathbf{s}(\phi) \\ f\mathbf{x}(\mathbf{x}/\boldsymbol{\theta}) &= C e^{-(\mathbf{x}-\mathbf{v})^H \mathbf{R}^{-1} (\mathbf{x}-\mathbf{v})} \end{aligned}$$

CRB for DOA Estimation

$$\ln f_{\mathbf{X}}(\mathrm{x}/\boldsymbol{\theta}) \;=\; \ln C + \frac{-\mathbf{x}^H\mathbf{x} + \alpha^*\mathbf{s}^H(\phi)\mathbf{x} + \alpha\mathbf{x}^H\mathbf{s}(\phi) - |\alpha|^2\,\mathbf{s}^H(\phi)\,\mathbf{s}(\phi)}{\sigma^2}$$

$$g(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\left[ae^{-jb}\mathbf{s}^H(\phi)\mathbf{x} + ae^{jb}\mathbf{x}^H\mathbf{s}(\phi) - a^2\mathbf{s}^H(\phi)\mathbf{s}(\phi)\right]$$

The CRB for the DOA Estimation problem is therefore,

$$\begin{aligned}\mathrm{var}(\phi) \;&\geq\; \left[\mathrm{E}\left(\frac{\partial^2 g}{\partial\phi^2}\right)\right]^{-1} \\ &\geq\; \frac{6\sigma^2}{|\alpha|^2 N(N^2-1)(kd)^2\sin^2\phi}\end{aligned}$$

DOA Estimation using MUSIC

MUSIC: **MU**ltiple **SI**gnal **C**lassification

$$\mathbf{x} = \mathbf{S}\alpha + \mathbf{n}$$

The matrix **S** is a *N* x *M* matrix of the *M* steering vectors

$$\begin{aligned} \mathbf{S} &= [\mathbf{s}(\phi_1)\ \mathbf{s}(\phi_2)\ \ldots,\ \mathbf{s}(\phi_M)] \\ \alpha &= [\alpha_1,\ \alpha_2\ \ldots\ \alpha_M]^T \end{aligned}$$

The correlation matrix of **x** can be written as

$$\begin{aligned} \mathbf{R} &= \mathrm{E}\left[\mathbf{x}\mathbf{x}^H\right] \\ &= \mathrm{E}\left[\mathbf{S}\alpha\alpha^H\mathbf{S}^H\right] + \mathrm{E}\left[\mathbf{n}\mathbf{n}^H\right] \\ &= \mathbf{S}\mathbf{A}\mathbf{S}^H + \sigma^2\mathbf{I} \\ &= \mathbf{R}_s + \sigma^2\mathbf{I} \end{aligned}$$

DOA Estimation using MUSIC

- R_s is *N* x *M* matrix with rank *M*
- It has *N* - *M* eigenvectors corresponding to zero eigenvalue
- Let $\boldsymbol{q}_m$ be such an eigenvector

$$
\begin{aligned}
\mathbf{R}_s \mathbf{q}_m &= \mathbf{S}\mathbf{A}\mathbf{S}^H \mathbf{q}_m = 0 \\
\mathbf{q}_m^H \mathbf{S}\mathbf{A}\mathbf{S}^H \mathbf{q}_m &= 0 \\
\mathbf{S}^H \mathbf{q}_m &= 0
\end{aligned}
$$

This implies that all *N* – *M* eigenvectors ($\boldsymbol{q}_m$) of R_s corresponding to the zero eigenvalues are *orthogonal* to all *M* signal steering vectors

DOA Estimation using MUSIC

- Let $\boldsymbol{q_m}$ be the N x (N – M) matrix of eigenvectors of $\boldsymbol{R_s}$
- The *pseudo - spectrum* is given by

$$P_{\text{MUSIC}}(\phi) = \frac{1}{\sum_{m=1}^{N-M} |\mathbf{s}^H(\phi)\mathbf{q}_m|^2} = \frac{1}{\mathbf{s}^H(\phi)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{s}(\phi)} = \frac{1}{||\mathbf{Q}_n^H\mathbf{s}(\phi)||^2}$$

- The eigenvectors making up $\boldsymbol{Q_n}$ are orthogonal to the signal steering vectors
- The denominator becomes zero when **ø** is a signal direction
- The estimated signal directions are the *M* largest peaks in the *pseudo – spectrum*

DOA Estimation using MUSIC

- Estimating the eigenvectors in $\boldsymbol{Q}_n$ from the eigenvectors of **R**
- For any eigenvector $\boldsymbol{q}_m \in \boldsymbol{Q}$,

$$\mathbf{R}_s \mathbf{q}_m = \lambda \mathbf{q}_m$$

$$\begin{aligned} \Rightarrow \mathbf{R}\mathbf{q}_m &= \mathbf{R}_s \mathbf{q}_m + \sigma^2 \mathbf{I} \mathbf{q}_m \\ &= (\lambda_m + \sigma^2) \mathbf{q}_m \end{aligned}$$

Therefore any eigenvector of is also an eigenvector of **R**
The corresponding eigenvalue $\lambda + \sigma^2$

If $\mathbf{R}_s = \mathbf{Q}\Lambda\mathbf{Q}^H$ then $\mathbf{R} = \mathbf{Q}\left[\Lambda + \sigma^2 \mathbf{I}\right]\mathbf{Q}^H$

DOA Estimation using MUSIC

- We can partition the eigenvector matrix **Q** into a signal matrix $\boldsymbol{Q_s}$ and noise subspace $\boldsymbol{Q_n}$
- The M columns of $\boldsymbol{Q_s}$ correspond to the M signal eigenvalues
- The $N - M$ columns of $\boldsymbol{Q_n}$ correspond to the noise eigenvalues
- The m-th signal eigenvalue is given by

$$\lambda_m + \sigma^2 = N|\alpha_m|^2 + \sigma^2$$

- By orthogonality of **Q**, $\boldsymbol{Q_s}$ is orthogonal to $\boldsymbol{Q_n}$

DOA Estimation using MUSIC

- We saw that all noise eigenvectors are orthogonal to the signal steering vectors
- This is the basis for MUSIC

$$P_{\mathrm{MUSIC}}(\phi) = \frac{1}{\sum_{m=M+1}^{N} |\mathbf{q}_m^H \mathbf{s}(\phi)|^2} = \frac{1}{\mathbf{s}^H(\phi)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{s}(\phi)}$$

where $\boldsymbol{q}_m$ is one of the $(N - M)$ noise eigenvectors

•If ***ø*** is equal to DOA of one of the signals, the denominator is zero
•MUSIC identifies the peaks of the function $\boldsymbol{P_{MUSIC}}(ø)$ as the directions of arrival

Root – MUSIC: Model Based Parameter Estimation

- In MUSIC accuracy is limited by the discretization at which $\boldsymbol{P_{MUSIC}}(\phi)$ is evaluated
- It requires human interaction or a comprehensive search algorithm to determine the peaks
- The Root - MUSIC method results directly in numeric values for the estimated directions
- It uses a *model* of the received signal as a function of the DOA
- The DOA, ϕ, is a *parameter* in this model.
- Based on this model and the received data the parameter is estimated

Root – MUSIC: Model Based Parameter Estimation

- We use steering vector as the model
- Assuming the receiving antenna is the linear array of equispaced, isotropic elements

$$\mathbf{s}(\phi) = [1,\, z,\, z^2,\, \ldots,\, z^{N-1}]^T$$

$$z = e^{jkd\cos\phi}$$

$$\Rightarrow \mathbf{q}_m^H \mathbf{s} = \sum_{n=0}^{N-1} q_{mn}^* z^n = q_m(z)$$

- The inner product of eigenvector $\boldsymbol{q_m}$ and the steering vector s(ø) is equivalent to a polynomial in z
- For $\mathbf{q}_m \perp \mathbf{s}(\phi)$ *m = (M+1),...,N,* we are looking for the *roots of a polynomial*

Root – MUSIC: Model Based Parameter Estimation

- The polynomial we use is given by

$$\begin{aligned} P_{\text{MUSIC}}^{-1}(\phi) &= \mathbf{s}^H(\phi)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{s}(\phi) \\ &= \mathbf{s}^H(\phi)\mathbf{C}\mathbf{s}(\phi) \end{aligned}$$

where $C = \boldsymbol{Q_n} \boldsymbol{Q_n}^{\boldsymbol{H}}$

$$\Rightarrow P_{\text{MUSIC}}^{-1}(\phi) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} z^n C_{mn} z^{-m} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} z^{(n-m)} C_{mn}$$

Taking $l = n - m$,

$$P_{\text{MUSIC}}^{-1}(\phi) = \sum_{l=-(N-1)}^{(N+1)} C_l z^l$$

$$C_l = \sum_{n-m=l} C_{mn}$$

Root – MUSIC: Model Based Parameter Estimation

- The polynomial obtained is of degree (2N-2)
- If **z** is a zero of the polynomial then so is $1/z^*$
- z and $1/z^*$ have the same phase and reciprocal magnitude hence both carry the same information
- One of these two lie within the unit circle
- Of the *(N – 1)* roots within the unit circle chose the M closest to the unit circle ($\boldsymbol{z_m}$, $m = 1,\ldots,M$)

Root – MUSIC: Model Based Parameter Estimation

- We obtained the directions of arrival using

$$\phi_m = \cos^{-1}\left[\frac{\Im \ln(z_m)}{kd}\right], \;\; m = 1, \ldots M$$

- As Root – MUSIC only worries about the phase of the roots, errors and the magnitude are irrelevant

THANK YOU