EE 602 – TERM PAPER PRESENTATION

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FOURIER SERIES BASED MODEL FOR STATISTICAL SIGNAL PROCESSING - CHONG YUNG CHI

ABSTRACT

- The goal of the paper is to present a parametric Fourier Series Based Model (FSBM) as an approximation of an unknown LTI system.
- Based on that FSBM, a minimum phase LPE filter for amplitude estimation of the LTI system together with CR bounds is presented.
- Then, two algorithms for finding the optimum LPE with finite Gaussian data are presented.

PRESENTATION OVERVIEW

- Introduction
- Algorithms to estimate amplitude and phase based on different decompositions of LTI system
- Comparison of results with conventional ARMA Models
- Conclusions
- queries

INTRODUCTION

- Assume an LTI system *h(n)*, driven by a random unknown signal *u(n)*, then

$$x(n) = u(n) * h(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k)$$

- The system function H(z) is often modeled as a parametric rational function(ARMA(p,q)) $H(z) = \frac{A(z)}{B(z)}$
- Also, it can be written as min max decomposition

$$H(z) = z^r . C(z) . D(z)$$

where

$$C(z) = \frac{\Pi_{k=1}^{M_i}(1 - a_k z^{-1})}{\Pi_{k=1}^{N_i}(1 - c_k z^{-1})} \qquad D(z) = \frac{\Pi_{k=1}^{M_0}(1 - b_k z)}{\Pi_{k=1}^{N_0}(1 - d_k z)}$$

Need of FSBM?

- To simplify the estimation problem?
- To find stability easily
- Since as in rational models we need to find poles and zeroes to find magnitude and phase of LTI system,it is difficult especially for systems with larger order but here we don’t need to find poles and zeroes we just have to find alpha and beta parameters.
- In the stabilty point of view also this problem arises.

REPRESENTATION OF FSBM

- Assume h(n) is a real LTI system with frequency response as H(w) = H*(-w) defined as

$$H(w) = \exp\{\sum_{i=1}^{p} \alpha_i \cos(i\omega) + \sum_{i=1}^{p} \beta_i \sin(i\omega)\}$$

This FSBM can be decomposed into two ways

- MG –PS form: $H(w) = H_{MG}(w) H_{PS}(w)$

$$H_{MG}(w) \quad \exp\left(\sum_{i=1}^{p} \alpha_i cosiw\right) \qquad H_{PS}(w) = j\exp\left(\sum_{i=1}^{p} \beta_i siniw\right)$$

where α_i , β are real and $p \geq 1$ and $q \geq 1$ and integers

- MP–AP form:

$$H(\omega) = H_{MG}(\omega)H_{AP}(\omega)$$

$$H_{MP}(\omega) = \exp\{\sum_{i=1}^{p}\alpha_i(\cos(i\omega) - j\sin(i\omega))\}$$

$$H_{AP}(\omega) = \exp\{j\sum_{i=1}^{\max(p,q)}(\alpha_i + \beta_i)\sin(i\omega)\}$$

where $\alpha_i + \beta_i = \gamma_i$

LPE FILTER

- A *conventional* p-th order LPE (causal FIR) filter looks like

$$A_p(z) = 1 + \sum_{i=1}^{p} a_i z^{-i}$$

and processes x(n) in a way so that the prediction error is

$$e(n) = x(n) * a_n = x(n) + \sum_{k=1}^{p} a_k x(n-k)$$

has minimum variance or average power. An optimum LPE filter in this case can be solved from the orthogonality principle

$$E[e(n)x(n-k)] = 0$$

- By estimation theory, for any unbiased estimates $\widehat{a_p}$ and $\widehat{\sigma^2}$ with given finite data, we can get the CR bounds for the covariance matrix

$$C_{\widehat{a_p},\widehat{\sigma^2}} = \frac{\sigma^2}{n}\begin{bmatrix} R_{xx}^{-1} & 0 \\ 0 & 2\sigma^2 \end{bmatrix}$$

where R_{xx} is the autocorrelation matrix, when x(n) is AR(p) Gaussian and pdf of x(n) is as follows:

$$P(x) = \prod_{i=p}^{n-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} e^2(n)\right)$$

where $$e(n) = x(n) + \sum_{i=1}^{p} a_k x(n-k)$$

FSBM FOR LPE Filter

- Let the pth order LPE filter (causal minimum phase IIR) be $V_p(n)$ with $V_p(0) = 1$

$$V_p(w) = \exp\left(\sum_{i=1}^{p} \alpha_i e^{-jwi}\right)$$

- Then the prediction error is

$$e(n) = x(n) * V_p(n) = x(n) + \sum_{k=1}^{p} V_p(k)x(n-k)$$

Theorem for Optimum LPE filter

- Given $x(n) = u(n) * h(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k)$ where

$u(n) \sim N(0, \sigma^2)$ and

H(w) is FSBM(p*,q), e(n) is the prediction error, then for any

$$p \geq p * \quad V_p\widehat{(w)} = 1/H_{MP}(w)$$

$$\min \left(E\left(e^2(n)\right) = E\left(u^2(n)\right) = \sigma^2\right.$$

PROOF OF THE ABOVE THEOREM

$$x(n) = u(n) \times h(n)$$

$$output\ PSD = input\ PSD \times |H(F)|^2$$

$$P_{xx}(f) = \sigma^2 |H(F)|^2$$

$$e(n) = x(n) \times a_n$$

$$E[e^2(n)] = \sigma^2 \sum_{n=0}^{\infty} |g(n)|^2$$

$$= \sigma^2 |g(0)|^2$$

$$= \sigma^2 \;------(1)$$

$$P_{ee}(\omega) = P_{xx}(\omega) \left|V_p(\omega)\right|^2$$

$$= \sigma^2 \left|H(\omega) \times V_p(\omega)\right|^2$$

$$= \sigma^2 \left|H_{MP}(\omega) \times V_p(\omega)\right|^2$$

$$= \sigma^2 |G(\omega)|^2$$

causal minimum phase filter, $g(0) = 1$

It holds only when $g(n) = \delta(n)$

$$G(\omega) = 1$$

$$g(n) = \delta(n)$$

$$G(\omega) = H_{MP}(\omega) \times V_p(\omega)$$

$$\frac{1}{H_{MP}(\omega)} = V_p(\omega)$$

with $E[e^2(n)] = \sigma^2$ from (1)

CR bounds

- For a FSBM(p,q) Gaussian Process , the CR bounds are given by

$$C_{\widehat{a_p},\widehat{\sigma^2}} = \frac{\sigma^2}{n}\begin{bmatrix} I & 0 \\ 0 & 2\sigma^2 \end{bmatrix}$$

- Note that the CR bounds associated with AR parameters depend on correlations of x(n), whereas those associated with α_i are uniform and independent of correlations of x(n). The CR bound associated with σ^2 is the same for both FSBM and AR model.

$$p(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\left(\frac{N}{2\sigma^2}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{I(\omega)}{|H(\omega)|^2}\,d\omega\right\}$$

$$I(\omega) = \frac{1}{N}|X(\omega)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\exp\{-j\omega n\}\right|^2$$

$$e(n) = x(n) \times a_n$$

$$P_{ee}(f) = P_{xx}(f)|A(F)|^2$$

$$= \sigma^2|H(F) \times A(F)|^2$$

$$= \sigma^2|G(F)|^2$$

$$\frac{P_{ee}(f)}{|G(F)|^2} = \sigma^2$$

$$\frac{\int R_{ee}(f)\, e^{-j2\pi f n} df}{|G(F)|^2} = \sigma^2$$

Algorithm 1

- Estimation of $H_{MP}(w)$ from a given finite data set based on the previous theorem.
- <u>Step 1</u>: Search for the minimum of the objective function

$$J(a_p) = \frac{\sum_{n=0}^{N-1} e^2(n)}{N}$$

and the associated optimum by a gradient-type iterative optimization algorithm (Fletcher Powell)

- Obtain the estimates $\hat{H}_{\mathrm{MP}}(\omega) = 1/\hat{V}_p(\omega) \quad (\text{or } \hat{\alpha}_p = -\hat{\mathbf{a}}_p)$

$$\widehat{\sigma^2} = J(\hat{\mathbf{a}}_p).$$

Estimation of order p of LPE filter

- Done by using the Akaike Information Criteria (AIC)

$$\mathrm{AIC}(k) = -2\ln p(\mathbf{x}; \hat{\alpha}_{k,\mathrm{AML}}, \widehat{\sigma^2}_{\mathrm{AML}}) + 2k$$

- The optimum estimate of p is one where AIC(k) is minimum for that p.
- Assumptions is that N>>p
- This is the simplified version after using AML approximation $\mathrm{AIC}(k) = N \ln \varepsilon(k) + 2k.$
- This criteria of estimating the order acts for both FIR as well as FSBM filters

Estimation of FSBM parameters

- Assumption: u(n) is non-Gaussian with non zero Mth order cumulant (x(n) is also non Guassian)
- Two algorithms are presented here which estimate the system amplitude using Algorithm 1 and system phase using a phase estimation algorithm that maximizes a single Mth order cumulant $|\hat{C}_M\{y(n)\}|$ of phase equalized (all pass filtered) data.
- Since the cumulant is a highly non linear function, we use Fletcher Powell algorithm to find the parameters.

Estimating the order q

- When q is unknown it must be estimated prior to the estimation of parameters.
- Here, we use an approach based on Cumulant Variation Rate (CVR(k)) defined as

$$\mathrm{CVR}(k) = \frac{|\eta(k) - \eta(k-1)|}{|\eta(k-1)|} \times 100\%$$

where η(k) is the maximum of $|\hat{C}_M\{y(n)\}|$ associated with the kth order all pass FSBM

- Optimum q is one for which CVR(k) is below a certain threshold

- Now we have estimated the orders p and q.
- Using that we present two algorithms to estimate the amplitude and phase parameters of the FSBM

Algorithm 2 (MG-PS)

- Step 1: Estimate $H_{MP}(w)$ and σ^2 from the Algorithm 1 to obtain $\hat{\alpha}_i$ from which we can obtain estimate of $H_{MG}(w)$
- Step 2: Find the optimum all pass FSBM $G_{AP}(w)$ such that $|\hat{C}_M\{y(n)\}|$ is maximum. Then obtain
- $\hat{\beta}_i = -\hat{\gamma}_i,\ i = 1, 2, \ldots, q,$ and $\hat{H}_{\mathrm{PS}}(\omega)$

since $\hat{G}_{\mathrm{AP}}(\omega) \simeq 1/H_{\mathrm{PS}}(\omega)$

Algorithm 3 (MP-AP)

- Step 1:Estimate $H_{MP}(w)$ and σ^2 from the Algorithm 1 to obtain $\widehat{\alpha_\iota}$ and obtain the maximum prediction error
  $e(n) \cong u(n) * h_{AP}(n)$
- Step 2: Find the optimum all pass FSBM G_{AP} (w) using a gradient type iterative algorithm such that M-th order cumulant or y(n) is maximum, $y(n) \cong u(n) * g_{AP}(n)$
  and the order of the all pass FSBM g is equal to max(p,q)
  Then, obtain $\widehat{\gamma_\iota} = -\widehat{\alpha_\iota} - \widehat{\beta_\iota}$ and $\widehat{H_{AP}}(w)$ since

$$\widehat{G_{AP}}(w) = \frac{1}{H_{AP}(w)}$$

Example

- Simulation results for the CR bounds associated with the amplitude parameters of the FSBM(p,q), estimation of p, and the performance evaluation of Algo 1.
- The driving input u(n) is assumed to be a zero mean white Gaussian random sequence and a non minimum phase FSBM(3,4) given by

$$H_{\mathrm{MG}}(\omega) = \exp\{1.1535\cos(\omega) - 0.4054\cos(2\omega) - 0.3138\cos(3\omega)\}$$

$$H_{\mathrm{PS}}(\omega) = \exp\{j[-0.9112\sin(\omega) + 0.5234\sin(2\omega) + 0.5290\sin(3\omega) + 0.2348\sin(4\omega)]\}$$

RESULTS

- AIC(k)

TABLE I

SIMULATION RESULTS OF PART 1 OF EXAMPLE 1. AIC(k), $k = 1 \sim 6$ AND $N = 512, 1024, 2048$ AND 4096

N \ p	1	2	3	4	5	6
512	120.0040	49.5771	3.8451	4.7642	5.8875	6.5956
1024	255.9041	107.3531	4.6679	5.5394	6.1014	7.1759
2048	507.7294	211.0764	9.6831	10.2516	11.0237	12.0000
4096	961.2921	365.2427	–24.4231	–23.2485	–22.0741	–20.4869

TABLE II

SIMULATION RESULTS OF PART 1 OF EXAMPLE 1. AVERAGES AND RMS ERRORS OF THIRTY INDEPENDENT AMPLITUDE PARAMETER ESTIMATES $\hat{\alpha}_i$, $i = 1 \sim 3$ USING ALGORITHM 1

	N	α_1 (1.1535)	α_2 (–0.4054)	α_3 (–0.3138)	$\frac{1}{\sqrt{N}}$
RMS error	512	0.0451	0.0414	0.0363	0.0442
	1024	0.0284	0.0322	0.0326	0.0312
	2048	0.0251	0.0209	0.0201	0.0221
	4096	0.0140	0.0152	0.0163	0.0156
Average	512	1.1712	–0.4011	–0.3114	
	1024	1.1487	–0.4071	–0.3238	
	2048	1.1500	–0.4083	–0.3185	
	4096	1.1504	–0.4076	–0.3130	

TABLE III

SIMULATION RESULTS OF PART 2 OF EXAMPLE 1. CVR(k) (%), $k = 1 \sim 8$ AND $N = 512, 1024, 2048$ AND 4096

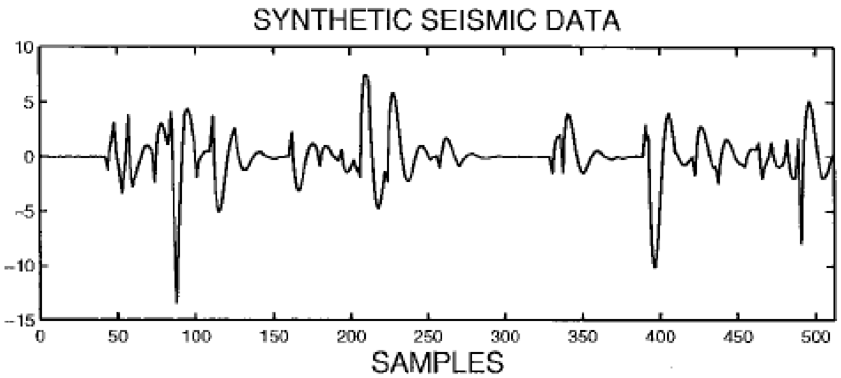
N \ q	1	2	3	4	5	6	7	8
512	17.40	77.13	37.08	3.01	0.49	0.47	0.39	0.28
1024	13.06	81.07	36.91	3.16	0.29	0.18	0.20	0.31
2048	14.21	83.61	34.03	2.88	0.19	0.13	0.01	0.22
4096	9.74	89.79	33.41	2.82	0.01	0.15	0.02	0.06

TABLE IV

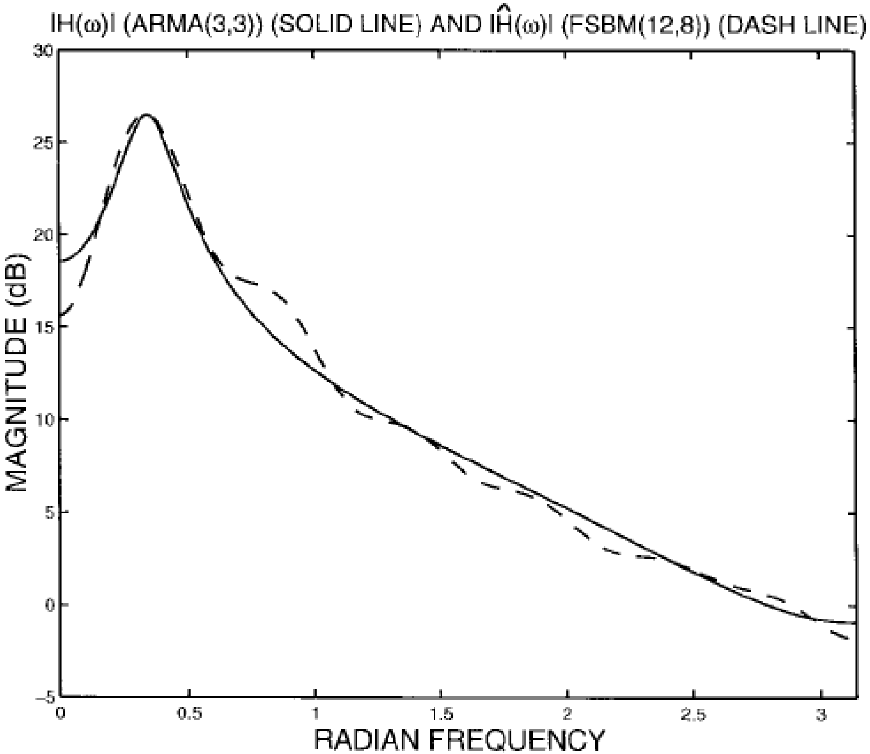
SIMULATION RESULTS OF PART 2 OF EXAMPLE 1. RMS ERRORS OF 30 INDEPENDENT AMPLITUDE PARAMETER ESTIMATES $\hat{\alpha}_i$, $i = 1 \sim 3$ AND PHASE PARAMETER ESTIMATES $\hat{\beta}_i$, $i = 1 \sim 4$ USING ALGORITHMS 2 AND 3 WITH $M = 3$, AND ALGORITHM 4 WITH $r = 2$ AND $m = 3$

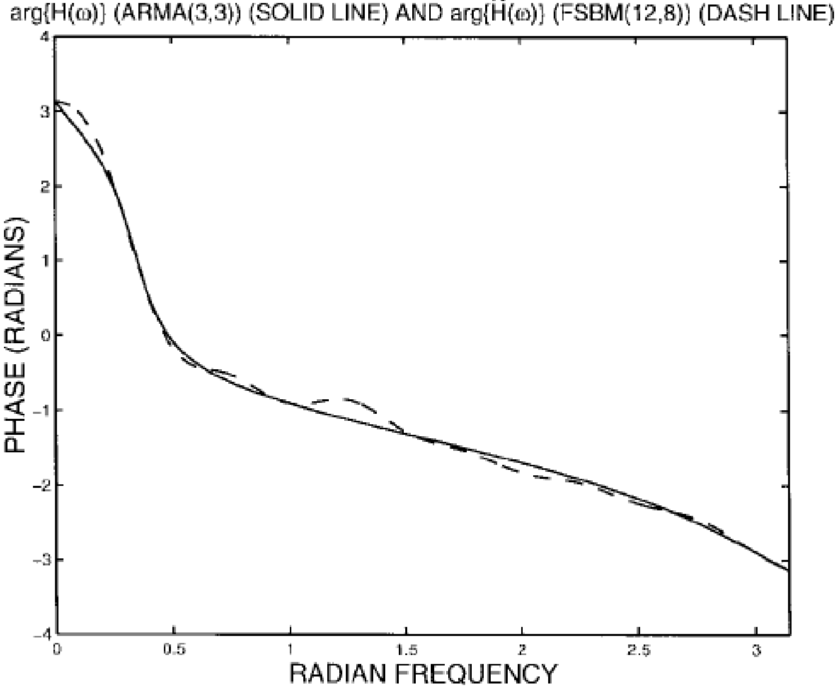
Algorithm	N	α_1 (1.1535)	α_2 (−0.4054)	α_3 (−0.3138)	β_1 (−0.9112)	β_2 (0.5234)	β_3 (0.5290)	β_4 (0.2348)
2	512	0.0403	0.0391	0.0423	0.1660	0.1044	0.0892	0.1030
	1024	0.0228	0.0263	0.0351	0.1290	0.0751	0.0772	0.0464
	2048	0.0172	0.0197	0.0235	0.1199	0.0650	0.0499	0.0482
	4096	0.0112	0.0146	0.0194	0.0407	0.0266	0.0354	0.0480
3	512	0.0403	0.0391	0.0423	0.0660	0.0735	0.0677	0.0523
	1024	0.0228	0.0263	0.0351	0.0441	0.0503	0.0492	0.0288
	2048	0.0172	0.0197	0.0235	0.0263	0.0269	0.0327	0.0311
	4096	0.0112	0.0146	0.0194	0.0211	0.0211	0.0217	0.0239
4	512	0.0679	0.0749	0.0617	0.0657	0.0771	0.0696	0.0559
	1024	0.0396	0.0423	0.0411	0.0431	0.0503	0.0485	0.0285
	2048	0.0283	0.0256	0.0308	0.0257	0.0266	0.0320	0.0314
	4096	0.0185	0.0186	0.0213	0.0210	0.0211	0.0216	0.0239

- AIC(k) data from the simulation results suggest that p=3, so it is good.
- For amplitude parameters , RMS error is nearly ewual to the square root of CR bounds, which justifies that Algorithm 1 is an AML estimator when the data is Gaussian.
- Comparing the RMS of the parameters for the two algorithms (2 and 3), we can draw conclusions.

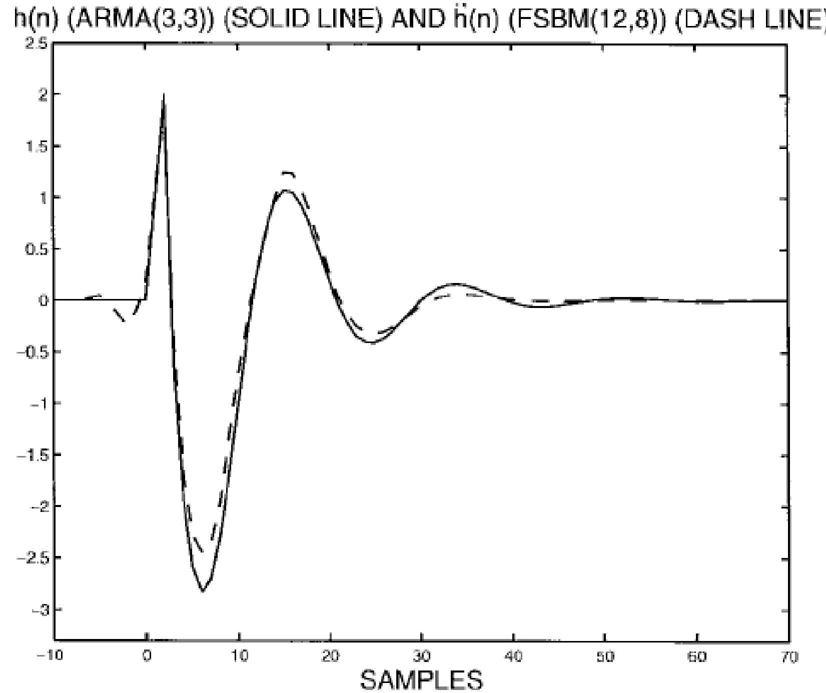

SYNTHETIC SEISMIC DATA
SAMPLES

(a)


|H(ω)| (ARMA(3,3)) (SOLID LINE) AND |Ĥ(ω)| (FSBM(12,8)) (DASH LINE)
MAGNITUDE (dB)
RADIAN FREQUENCY


arg{H(ω)} (ARMA(3,3)) (SOLID LINE) AND arg{Ḧ(ω)} (FSBM(12,8)) (DASH LINE)
PHASE (RADIANS)
RADIAN FREQUENCY

(c)


h(n) (ARMA(3,3)) (SOLID LINE) AND ḧ(n) (FSBM(12,8)) (DASH LINE)
SAMPLES

(d)

- the approximation of FSBM(p,q) to the giventrue ARMA system is never perfect for this case, even when $p=q=\infty$.
- This also implies that the FSBM is merely a stable approximation to any arbitrary LTI systems, regardless of pole and zero locations.

Conclusions

- As it turns out from simulations performed, Algorithm 2 and 3 perform nearly the same for amplitude parameter estimation. (since same method is used)
- For phase estimation, Algorithm 3(MP-AP) performs better than 2 (MG-PS)
- So, overall, Algorithm 3 is preferred.
- FSBM is useful for frequency domain implementations such as deconvolution and channel equalization, system identification, speech coding and compression, time delay estimation, signal detection and classification.

THANK YOU!!