Matched Filter for Binary and Mary Spaces

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Introduction

The problem here pertains to detection of known signal in Gaussian noise. Here we assume noise to be Gaussian therefore the resultant test statistic is linear function of the data. The detector evolved from the above assumptions is termed as Matched filter.

Development of detectors

The criterion used for detection is Neyman–Pearson(NP)- criterion. However Bayesian risk criterion could also be used, only threshold and detection performance will differ.

The detection problem is to distinguish between the hypotheses.

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] & n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = s[n] + w[n] & n = 0, 1, \ldots, N-1 \end{aligned}$$

Where s[n] is the known deterministic signal and w[n] is WGN with variance σ^2.

The NP Detector decides H1 if the likelihood ratio exceeds the threshold or,

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$$

where $\mathbf{x} = [x[0]\, x[1] \ldots x[N-1]]^T$. Since

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n])^2\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right] \end{aligned}$$

we have

$$L(\mathbf{x}) = \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2 - \sum_{n=0}^{N-1}x^2[n]\right)\right] > \gamma.$$

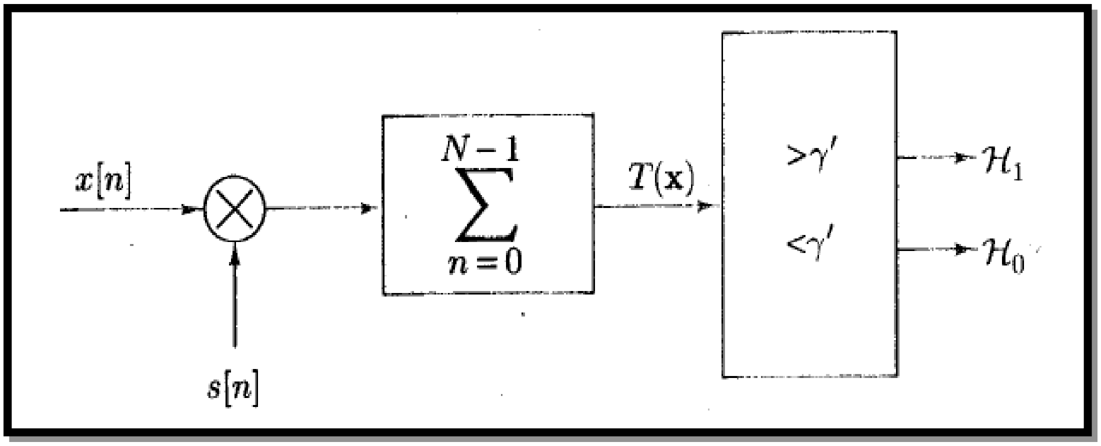
We decide H1 if ,

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'.$$

Where new threshold γ' is given by,

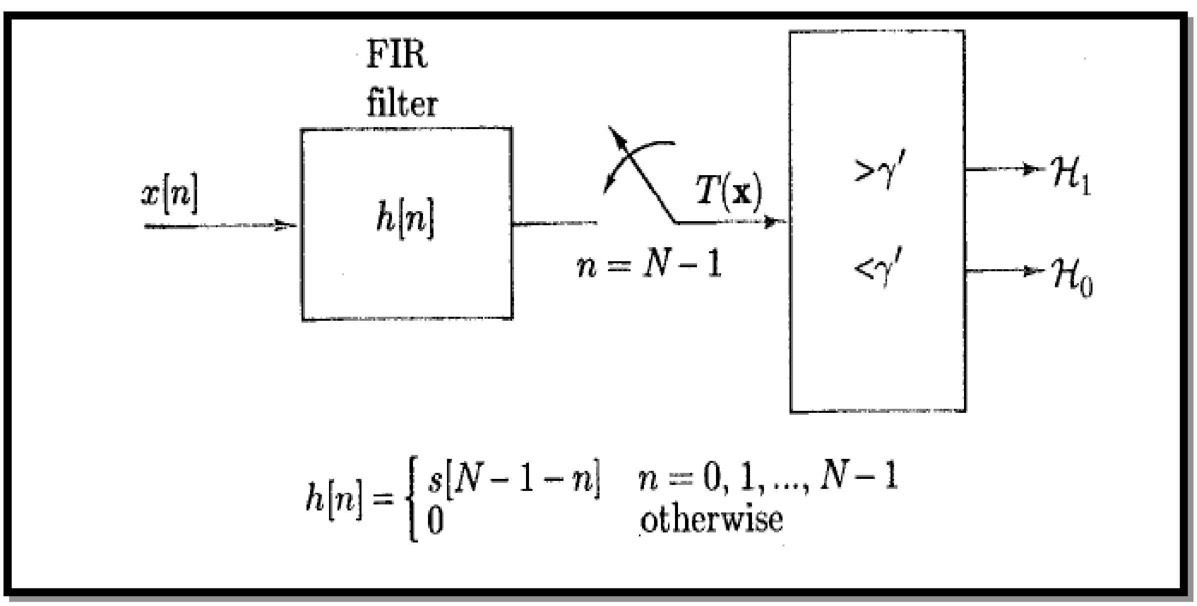
$$\sigma^2 \ln\gamma + \frac{1}{2}\sum_{n=0}^{N-1} s^2[n].$$

The above mentioned detector is referred to as a correlator or replica correlator since we correlate the received data with a replica of the signal. The implementation is shown below.



Alternatively we can implement the above correlator using FIR filter on the data which is termed as matched filter.

The implementation of matched filter is shown below:-



Where x[n] is the input to the FIR filter with impulse response h[n]. Where h[n] is non zero for n=0 , 1, … N-1.

The output at time n $\geq$ 0 is:

$$y[n] = \sum_{k=0}^{n} h[n-k]x[k]$$

As we know:

$$h[n] = s[N-1-n] \qquad n = 0, 1, \ldots, N-1$$

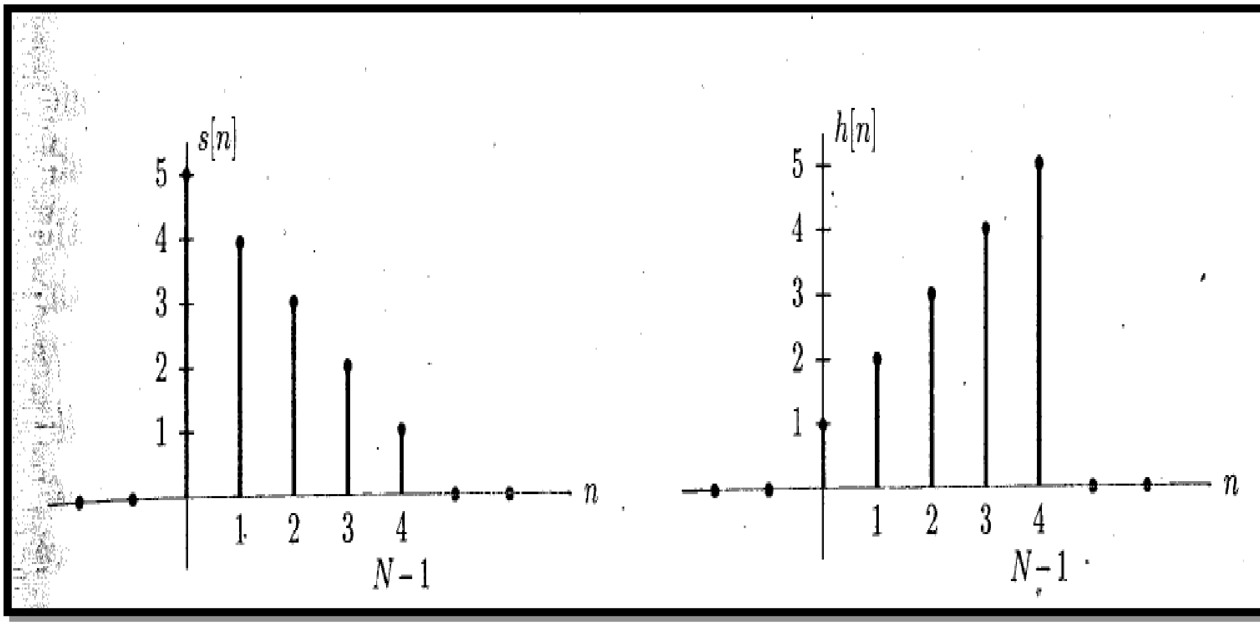
$$y[n] = \sum_{k=0}^{n} s[N-1-(n-k)]x[k].$$

Now the output of the filter at time n= N-1 is ,

$$y[N-1] = \sum_{k=0}^{N-1} s[k]x[k]$$

The filter impulse response h[n] is matched to the signal and is obtained by flipping s[n] about n=0 and shifting to the right by N-1 samples.

Example of matched filter response for N=5



The signal output attains a maximum at the sampling time n=N-1. However is the signal does not begin at n=0 poor detection performance may be obtained thus for signal with unknown arrival time we can not use the matched filter in its present form.

<u>Properties of Matched filter</u>

1. In absence of noise the matched filter output is just the signal energy.
2. The matched filter maximizes the SNR at the output of an FIR filter which is given by

$$\eta = \frac{E^2(y[N-1];\mathcal{H}_1)}{\mathrm{var}(y[N-1];\mathcal{H}_1)}$$

$$= \frac{\left(\sum_{k=0}^{N-1} h[N-1-k]s[k]\right)^2}{E\left[\left(\sum_{k=0}^{N-1} h[N-1-k]w[k]\right)^2\right]}$$

Performance of Matched Filter

To determine the performance of the filter we derive P_D for a given P_{FA}.

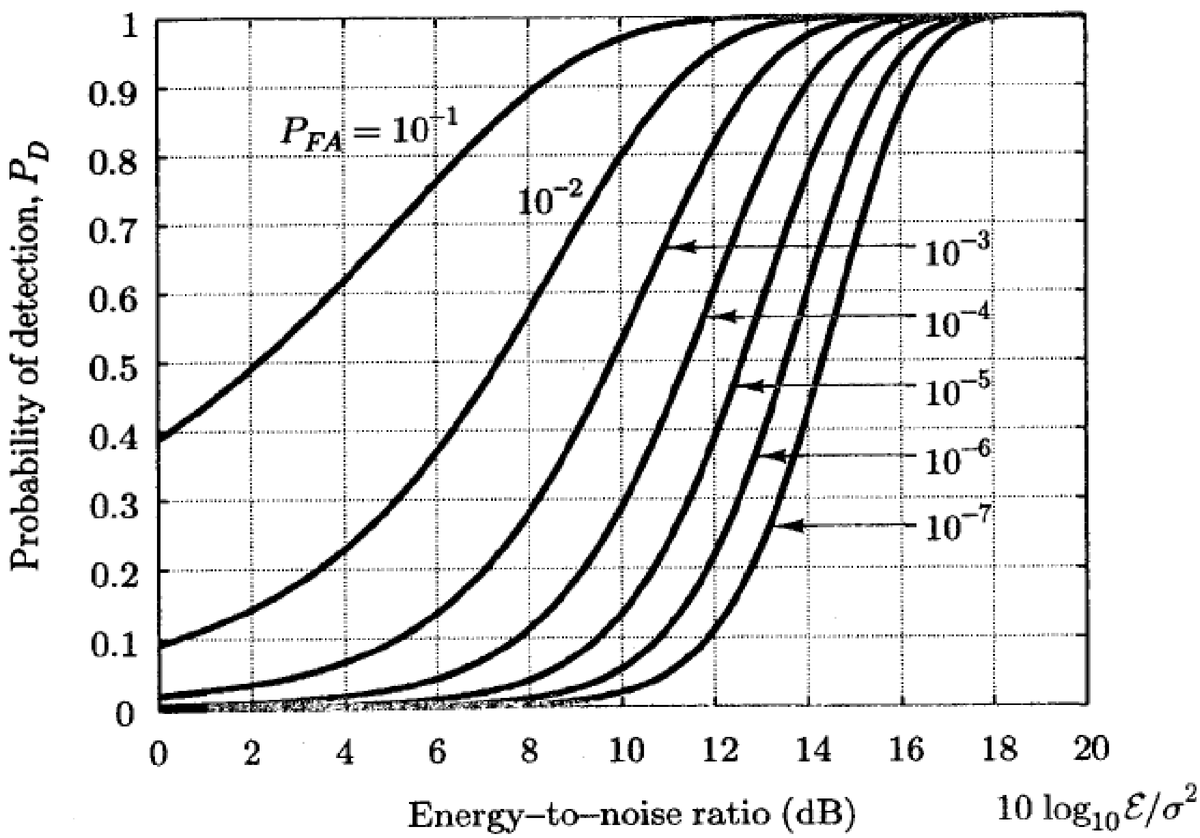
The relation between P_D and P_{FA} is given by:

$$P_D = Q\left(\frac{\sqrt{\sigma^2\mathcal{E}}Q^{-1}(P_{FA})}{\sqrt{\sigma^2\mathcal{E}}} - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right).$$

The Key parameter is the SNR (ϵ/σ^2) at the matched filter output, as SNR increases P_D increases. The shape of the signal does not affect the detection performance.

Detection performance of matched filter against ENR (d B) is given:-



Generalized Matched filter

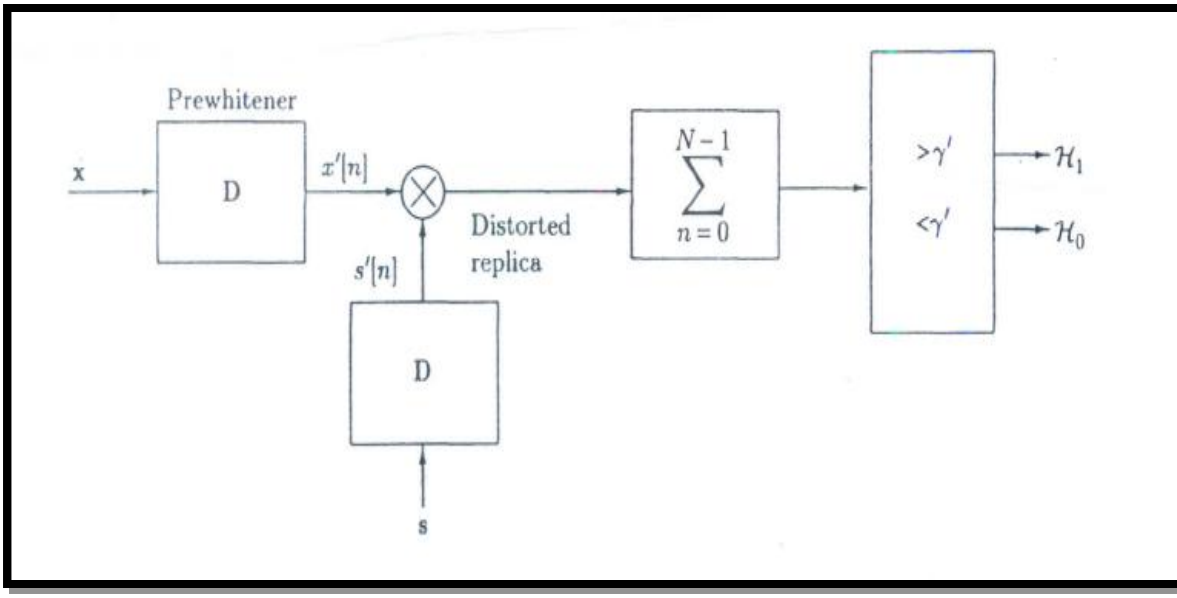
We have seen that Matched filter is optimum only when the Noise is WGN. When Noise is correlated (colored noise) but WSS then we can pre-whiten the received data and signal to achieve the better performance .

We decide H1 if

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} > \gamma'.$$

- Where C is the Correlation matrix.
- Since C is positive definite then it can be shown that C^{-1} exists and is also positive definite therefore can be factored as $C^{-1} = D^T D$
- The Test Statistic becomes $T(x) = X^T D^T D$
- Where D is pre-whitening Matrix.

The generalized matched filter as pre whitener plus replica correlator (matched filter) is given below:-



In this case the matched performance equation is given by:

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

In case of Generalized Matched filter the P_D increases monotonically with $s^T C^{-1} s$ not with SNR $(Є/\sigma^2)$. Hence signal can be designed to maximize $s^T C^{-1} s$ and hence P_D .Signal shape is important in this case.

Detection of multiple signals

Case – 1 "Binary Case "

In communication systems we transmit one of M signals .The signals are known to the receiver but

receiver has to decide which one. For Binary case we consider M=2.
For this case we have the following hypothesis testing problem.

H_0 : x(n) = s_0(n)+w(n) ; n= 0,1,2…….,N-1

H_1 : x(n) = s_1(n)+w(n) ; n= 0,1,2…….,N-1

Where s_0(n) and s_1(n) are known deterministic signals and w[n] is WGN with variance σ^2

Here each type of error (Type I and Type II) is equally undesirable , the minimum probability of error criterion is chosen. We decide H1 if

$$\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \gamma = \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = 1$$

Here we have assumed equal prior probabilities of transmitting S_0 and S_1.We choose the hypothesis having larger conditional likelihood.

$$p(\mathbf{x}|\mathcal{H}_i) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s_i[n])^2\right]$$

We decide Hi for which the $D_i^{\,2}$,

$$D_i^2 = \sum_{n=0}^{N-1} (x[n] - s_i[n])^2$$

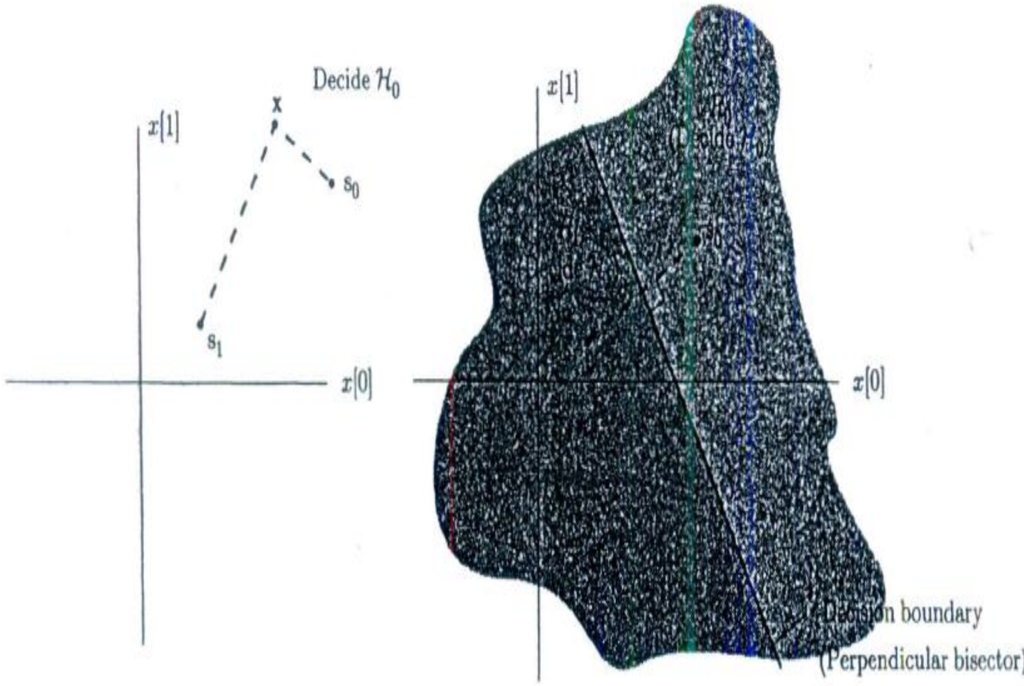
This is referred to as a minimum distance receiver.

If we consider the data and signal samples in R^N then

$$\begin{aligned} D_i^2 &= (\mathbf{x} - \mathbf{s}_i)^T(\mathbf{x} - \mathbf{s}_i) \\ &= ||\mathbf{x} - \mathbf{s}_i||^2 \end{aligned}$$

We choose the hypothesis whose signal vector is closest to X.

If we consider N=2 The decision region is decided by dividing the plane in to two regions which are separated by the perpendicular bisector of the line segment joining the two signal vector as shown below:-



The minimum distance receiver can also be expressed in a more familiar form

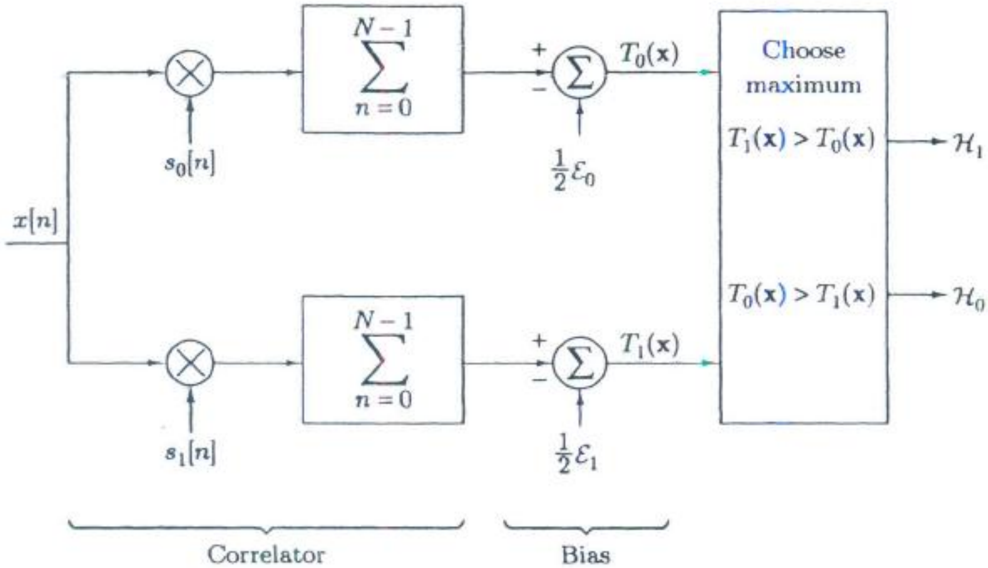
$$D_i^2 = \sum_{n=0}^{N-1} x^2[n] - 2\sum_{n=0}^{N-1} x[n]s_i[n] + \sum_{n=0}^{N-1} s_i^2[n]$$

we decide $\mathcal{H}_i$ for which

$$\begin{aligned} T_i(\mathbf{x}) &= \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\sum_{n=0}^{N-1} s_i^2[n] \\ &= \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\mathcal{E}_i \end{aligned}$$

is maximum.

The minimum distance receiver for binary signal detection if given below:-



Performance analysis for Binary Case:-

We determine the Pe for the ML receiver for equal signal probabilities which is given by :-

$$
\begin{aligned}
P_e &= \frac{1}{2}\left[P(\mathcal{H}_1|\mathcal{H}_0) + P(\mathcal{H}_0|\mathcal{H}_1)\right] \\
&= \frac{1}{2}\left[\Pr\{T_1(\mathrm{x}) > T_0(\mathrm{x})|\mathcal{H}_0\} + \Pr\{T_0(\mathrm{x}) > T_1(\mathrm{x})|\mathcal{H}_1\}\right] \\
&= \frac{1}{2}\left[\Pr\{T_1(\mathrm{x}) - T_0(\mathrm{x}) > 0|\mathcal{H}_0\} + \Pr\{T_0(\mathrm{x}) - T_1(\mathrm{x}) > 0|\mathcal{H}_1\}\right].
\end{aligned}
$$

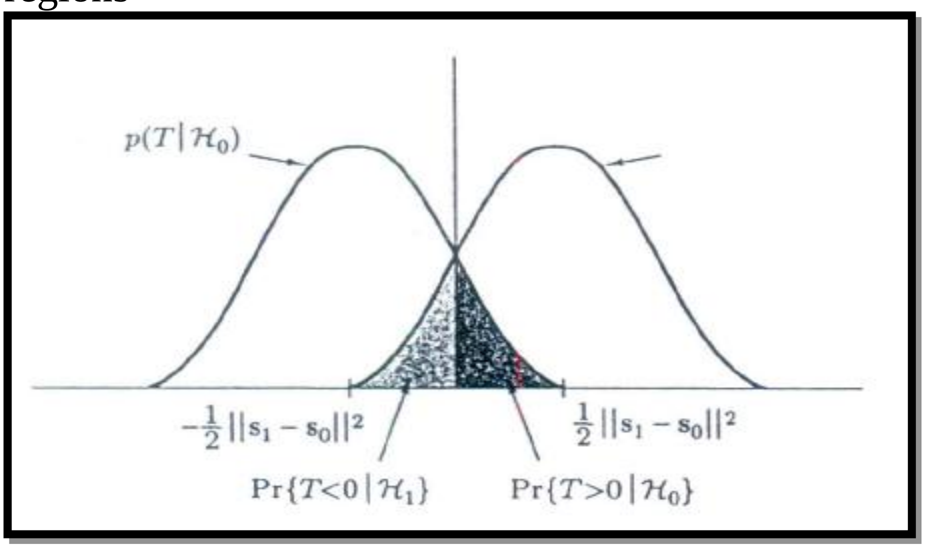
Let $T(\mathbf{x}) = T_1(\mathbf{x}) - T_0(\mathbf{x})$. Then,

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n](s_1[n] - s_0[n]) - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)$$

The test static is Gaussian random variable with :-

$$T \sim \mathcal{N}(-\tfrac{1}{2}||\mathbf{s}_1 - \mathbf{s}_0||^2, \sigma^2||\mathbf{s}_1 - \mathbf{s}_0||^2)$$

The errors for binary signal detection is provided by shaded regions



The error are same because of inherent receiver symmetry. The probability of error is given by :-

$$P_e = \Pr\{T(\mathbf{x}) > 0 | \mathcal{H}_0\}$$

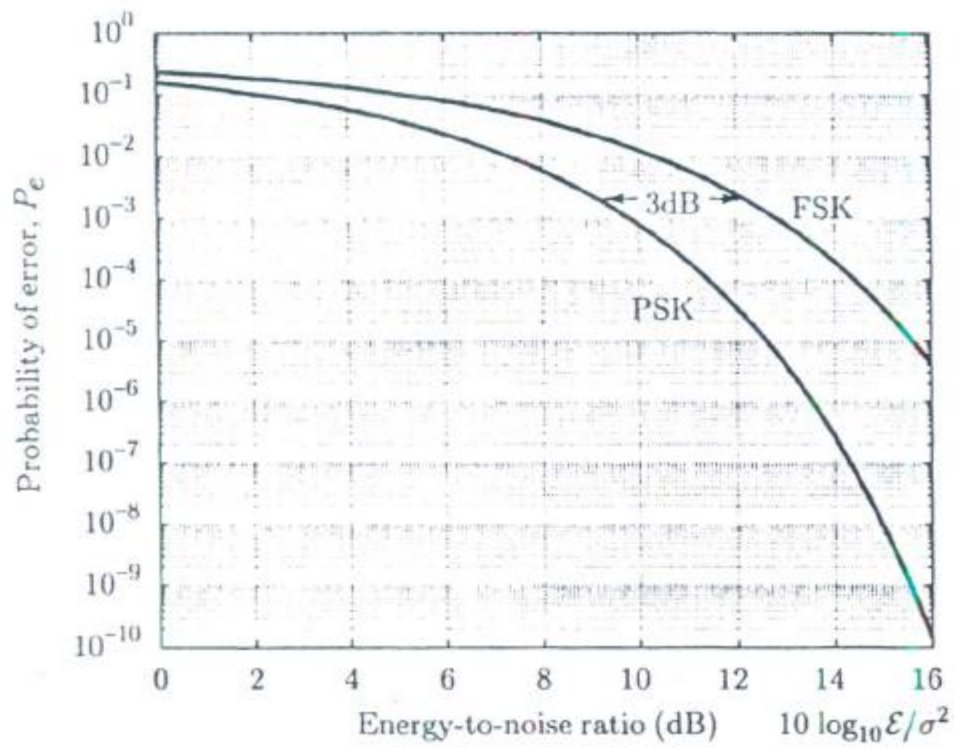
$$P_e = Q\left(\frac{\frac{1}{2}||\mathbf{s}_1 - \mathbf{s}_0||^2}{\sqrt{\sigma^2||\mathbf{s}_1 - \mathbf{s}_0||^2}}\right)$$

or finally

$$P_e = Q\left(\frac{1}{2}\sqrt{\frac{||\mathbf{s}_1 - \mathbf{s}_0||^2}{\sigma^2}}\right).$$

As $||s_1 - s_0||$ increases , Pe decreases as expected. However there is limitation in average power due to FCC regulations or system constraints. Average energy assuming equal prior probabilities.

Performance of typical binary signaling schemes (FSK, PSK) are plotted below :



Case 2 : "M-ary Case"

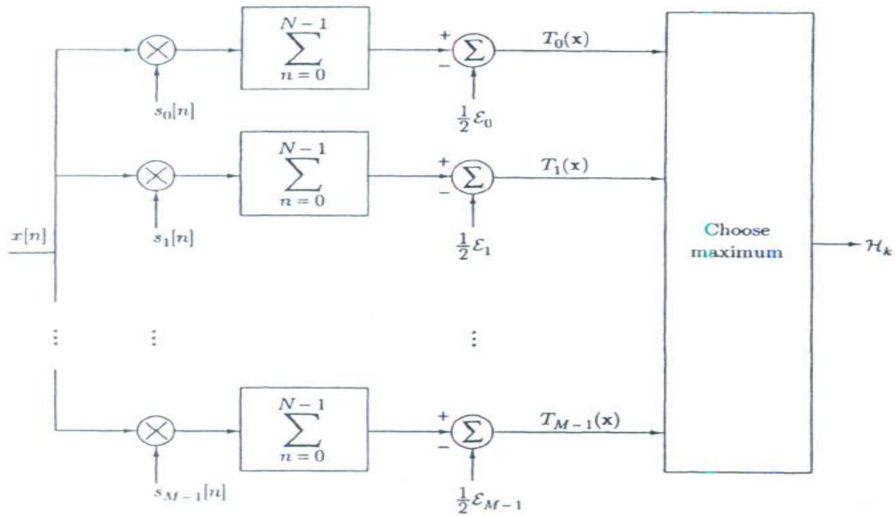
Now if we transmit one of M signals with equal prior probabilities. The optimal receiver is again a minimum distance receiver and we choose H_k

$$T_k(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s_k[n] - \frac{1}{2}\mathcal{E}_k$$

$T_k(x)$ is the maximum test static of

$$\{T_0(\mathbf{x}), T_1(\mathbf{x}), \ldots, T_{M-1}(\mathbf{x})\}$$

The Optimal receiver is



An error occurs is any of the M-1 test statistic exceeds the one associated with true hypothesis. If we consider signal to be orthogonal , then the statistics will be independent. Since joint Gaussian random variables with orthogonality assumption are uncorrelated hence independent. If the signal energies are same, an error is committed, if Hi is the true hypothesis but Ti is not maximum. Therefore

$$P_e = \sum_{i=0}^{M-1} \Pr\{T_i < \max(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{M-1}) | \mathcal{H}_i\} P(\mathcal{H}_i).$$

By symmetry all of the conditional probabilities in the above sum are same and hence

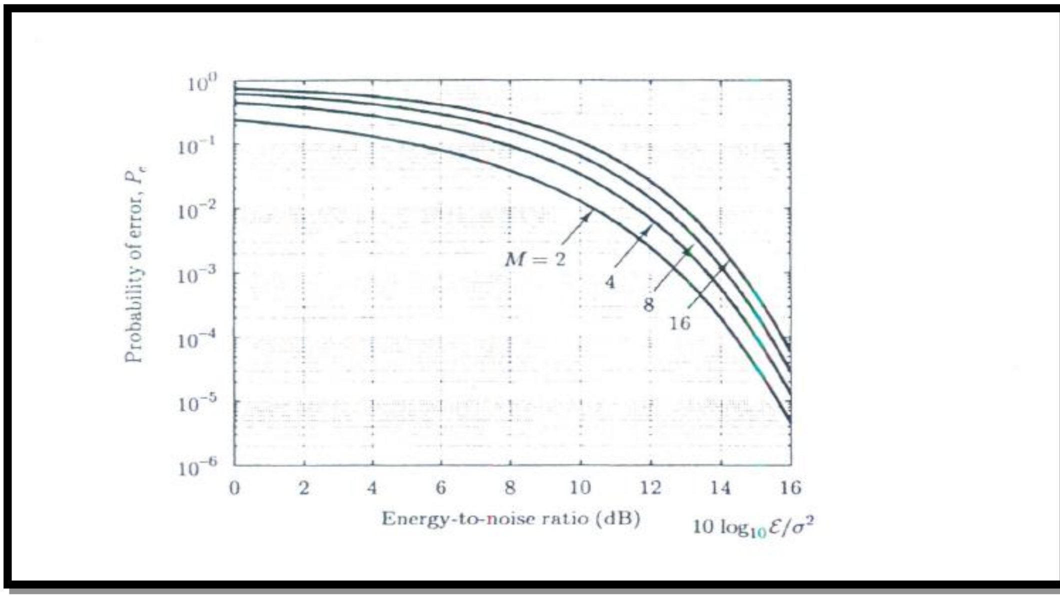
$$P_e = \Pr\{T_0 < \max(T_1, T_2, \ldots, T_{M-1}) | \mathcal{H}_0\}.$$

Therefore,

$$\begin{aligned} P_e &= 1 - \Pr\{T_0 > \max(T_1, T_2, \ldots, T_{M-1}) | \mathcal{H}_0\} \\ &= 1 - \Pr\{T_1 < T_0, T_2 < T_0, \ldots, T_{M-1} < T_0 | \mathcal{H}_0\} \\ &= 1 - \int_{-\infty}^{\infty} \Pr\{T_1 < t, T_2 < t, \ldots, T_{M-1} < t | T_0 = t, \mathcal{H}_0\} p_{T_0}(t) dt \\ &= 1 - \int_{-\infty}^{\infty} \prod_{i=1}^{M-1} \Pr\{T_i < t | \mathcal{H}_0\} p_{T_0}(t) dt \end{aligned}$$

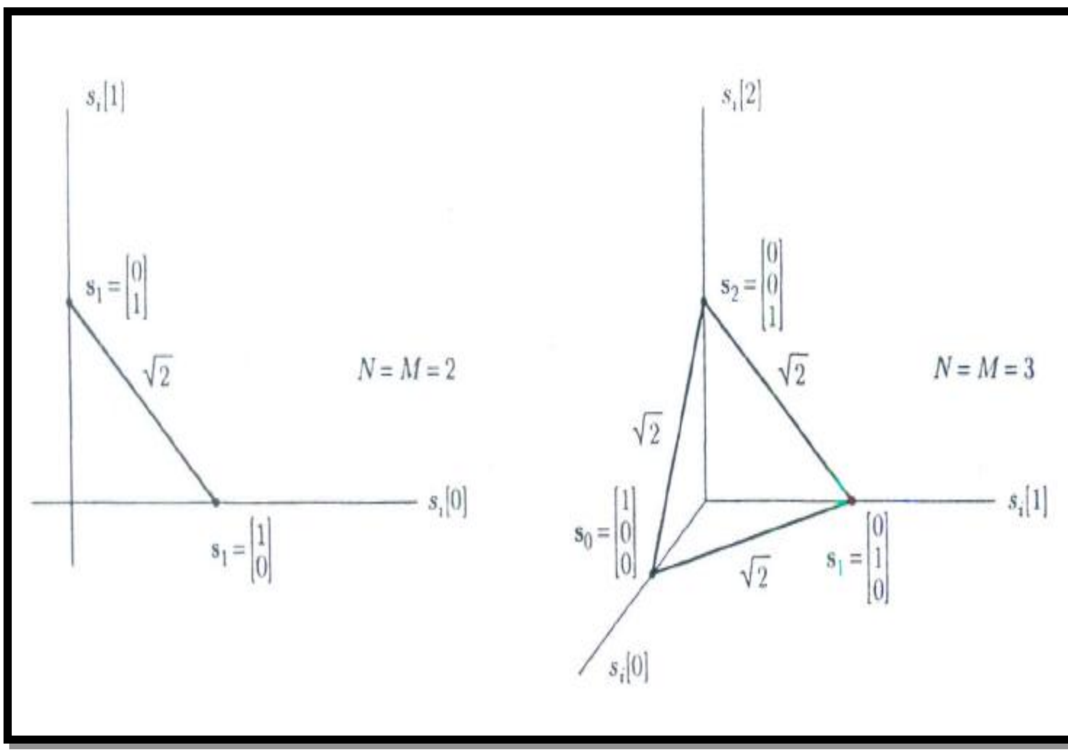
$$P_e = 1 - \int_{-\infty}^{\infty} \Phi^{M-1}(u) \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(u - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)^2\right] du$$

The expression shows that Pe depends on the ENR (Є/σ2).This dependence is plotted for various values of M :-



Consider the cases of M=N=2 and M=N=3 orthogonal signals.

The illustration of increasing Pe with increase in number of M is shown below:-



Here the distance between the signals are the same for M=2 and M=3 ,each signal has energy Є=1.As M increases we must choose from among a larger set of signals and therefore , Pe must increase with M.