1. We observe N IID samples from the PDFs:

   (a) Gaussian -
   $$p(x;\mu) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(x-\mu)^2\right] \tag{1}$$

   (b) Exponential -
   $$p(x;\mu) = \begin{cases} \lambda\exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases} \tag{2}$$

   In each case find the MLE of the unknown parameter and be sure to verify that it indeed maximizes the likelihood function. Do the estimators make sense?

2. Asymptotic results can sometimes be misleading and so should be carefully applied. As an example, for two unbiased estimators of θ the variances are given by
   $$\mathrm{var}(\hat{\theta}_1) = \frac{2}{N} \tag{3}$$
   $$\mathrm{var}(\hat{\theta}_2) = \frac{1}{N} + \frac{100}{N^2} \tag{4}$$
   Plot the variances versus N to determine the better estimator.

3. A formal definition of the consistency of an estimator is given as follows. An estimator $\hat{\theta}$ is consistent if, given an $\epsilon > 0$,
   $$\lim_{N\to\infty} Pr\left\{|\hat{\theta}-\theta| > \epsilon\right\} = 0. \tag{5}$$
   Prove that the sample mean is a consistent estimator for the problem of estimating a DC level A in white Gaussian noise of known variance. Hint: Use Chebychev's inequality.

4. If we observe N IID samples from a Bernoulli experiment (coin toss) with the probabilities
   $$\Pr\{x[n] = 1\} = p \tag{6}$$
   $$\Pr\{x[n] = 0\} = 1 - p \tag{7}$$
   find the MLE of p.

5. For N IID observations from the PDF $\mathcal{N}(A, \sigma^2)$, where A and σ^2 are both unknown, find the MLE of the SNR $\alpha = A^2/\sigma^2$.

6. In fitting a line through experimental data we assume the model
   $$x[n] = A + Bn + w[n],\ -M \le n \le M \tag{8}$$
   where $w[n]$ is WGN with variance σ^2. If we have some prior knowledge of the slope B and intercept A such as
   $$\begin{bmatrix} \mathrm{A} \\ \mathrm{B} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix}\right) \tag{9}$$
   find the MMSE estimator of A and B as well as the minimum Bayesian MSE. Assume that A, B are independent of $w[n]$. Which parameter will benefit most from the prior knowledge?

7. For the posterior PDF

$$p(\theta|x) = \frac{\epsilon}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\theta - x)^2\right] + \frac{1-\epsilon}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\theta + x)^2\right]. \tag{10}$$

   Plot the PDF for $\epsilon = 1/2$ and $\epsilon = 3/4$. Next, find the MMSE and MAP estimators for the same values of ϵ.

8. The data $x[n] = A + w[n]$ for $n = 0, 1, ..., N-1$ are observed. The unknown parameter A is assumed to have the prior PDF

$$p(A) = \begin{cases} \lambda\exp(-\lambda A) & A > 0 \\ 0 & A < 0 \end{cases} \tag{11}$$

   where $\lambda > 0$, and $w[n]$ is WGN with variance σ^2 and is independent of A. Find the MAP estimator of A.

9. Derive the fisher information matrix for the parameters (A, ϕ, w, σ^2) in the model

$$x_t = s_t + n_t \quad ; \quad t = 0, 1, ..., N-1 \tag{12}$$

$$s_t = A\,cos(wt - \phi) \quad ; \quad n_t : N[0, \sigma^2] \tag{13}$$

   Plot your variance bounds versus SNR, parameterized by N and vice versa.[Ref. secion 6.12 of Schafer]

10. Let $w = \theta^T\theta$ and assume $x : N[H\theta, R]$. Find the fisher information matrix for w.

11. Let $X = (X_1, X_2, ..., X_N)$ denote a random sample of exponential random variables with unknown parameter θ:

$$f_\theta(x_n) = \frac{1}{\theta}e^{-x/\theta} \quad ; \quad 0 \leq x < \infty \quad , \quad 0 \leq \theta < \infty \tag{14}$$

   Compute the fisher information matrix. Is the ML estimate unbiased ? minimum variance ? efficient?