1. Design a perfect detector for the problem

$$\mathcal{H}_0 : x[0] \sim \mathcal{U}[-c, c]$$

$$\mathcal{H}_1 : x[0] \sim \mathcal{U}[1-c, 1+c]$$

   where $c > 0$ and $\mathcal{U}[a, b]$ denotes a uniform PDF on the interval $[a, b]$, by choosing $c$. A perfect detector has $P_{FA} = 0$ and $P_D = 1$.

2. Prove that the ROC is a concave function over the interval [0,1]. A concave function is one for which

$$\alpha g(x_1) + (1-\alpha) g(x_2) \leq g(\alpha x_1 + (1-\alpha) x_2)$$

   for $0 \leq \alpha \leq 1$ and any two points $x_1$ and $x_2$. To do so consider two points on the ROC $(p_1, P_D(p_1))$ and $(p_2, P_D(p_2))$ and find $P_D$ for a randomized test. A randomized test first flips a coin with $\Pr\{\text{head}\} = \alpha$. If the outcome is a head, we employ the detector whose performance is $(p_1, P_D(p_1))$. Otherwise, we employ the detector whose performance is $(p_2, P_D(p_2))$. We decide $\mathcal{H}_1$ if the chosen decides $\mathcal{H}_1$. Hint: For a given $P_{FA}$ the detection performance of the randomized detector must be less than or equal to that of the NP detector.

3. Find the MAP decision rule for

$$\mathcal{H}_0 : x[0] \sim \mathcal{N}(0, 1)$$

$$\mathcal{H}_1 : x[0] \sim \mathcal{N}(0, 2)$$

   if $P(\mathcal{H}_0)$ =1/2 and also if $P(\mathcal{H}_0)$ =3/4. Display the decision regions in each case and explain.

4. A general $M$-ary PAM communication system transmits one of $M$ DC levels. Let the levels be $\{0, \pm A, \pm 2A, ..., \pm(M-1)A/2\}$ for $M$ odd. The received data $x[n]$ for $n$= 0,1,...,$N$-1 will be one of the DC levels embedded in WGN with variance $\sigma^2$. Using the concept of sufficient statistics from Problem 3.19 find the ML detector. Then, show that the minimum $P_e$ is

$$P_e = \frac{2M-2}{M} Q\left(\sqrt{\frac{NA^2}{4\sigma^2}}\right)$$

   for equal prior probabilities.

5. We wish to detect a damped exponential $s[n] = Ar^n$, where $A$ is unknown and r is known $(0 < r < 1)$, in WGN with known variance $\sigma^2$. Based on $x[n]$ for $n$= 0,1,...,$N$-1 show that the GLRT decides $\mathcal{H}_1$ if $\hat{A}^2 > \gamma'$, where $\hat{A}$ is MLE of $A$.

6. Consider the parameter test

$$\mathcal{H}_0 : \theta = \theta_0$$

$$\mathcal{H}_1 : \theta \neq \theta_0$$

   We observe the IID samples $x[n]$ for $n$= 0,1,...,$N$-1. If the PDF can be factored as

$$p(\mathbf{x}; \theta) : g(T(\mathbf{x}), \theta) h(\mathbf{x})$$

   then $T(\mathbf{x})$ is a sufficient statistics for $\theta$ ([Kay-I 1993,pp.104-105]). Assume that $T(\mathbf{x})$ is a sufficient statistics for $\theta$ and find the GLRT statistics for this problem to show that it is a function of only the sufficient statistics. Hint: Recall that $h(\mathbf{x}) \geq 0$ for all $\mathbf{x}$.

7. We wish to detect a sinusoid $s[n] = a\ cos 2\pi f_0 n + b\ sin 2\pi f_0 n$ for $n$= 0,1,...,$N$-1 embedded in WGN with known variance $\sigma^2$. The frequency is $f_0 = k/N$ for $k \in \{1, 2, ..., N/2 - 1\}$. If $a$ and $b$ are unknown, determine the GLRT test. Also, determine the detection performance and interpret the non centrality parameter. Hint: Note that the data can be put in the form of the linear model so that the results of Problems 6.15 and 6.17 (S.M.Kay,volume II,Fundamentals of SSP Detection theory) can be applied. Also, the columns of **H** are orthogonal.

8. *ROC curves.* Consider the false alarm probability $P_{FA} = 1 - \Phi(z)$ and the detection probablity $P_D = 1 - \Phi(z - d)$. Compute and plot $P_D$ versus $P_{FA}$ for several representative values of $d$. what values of $d$ is required to achieve $(P_D = 0.99, P_{FA} = 0.01)$? compute and plot $P_D$ versus $d$ for several representative values of $P_{FA}$. Fix $P_D$ at 0.5 and compute and plot the threshold $\eta$ versus the false alarm probability $P_{FA}$.

9. Lets send the binary digit $i\ \epsilon\ (0, 1)$ by transmitting this signal on the baud interval $n = 0, 1, ..., N - 1$

$$s_n = cos(2\pi m_i n/N) \quad ; \quad m_i\ (integer) \tag{1}$$

This is called a frequency shift keyed (FSK) symboling scheme. Observations are $y_n = s_n + n_n$ with the $n_n$ i.i.d $N[0, \sigma^2]$ random variables.

   (a) Derive an optimum detector for this problem. Illustrate the detector and completely charecterize its performance when $\alpha$ is required to equal to $1 - \beta$
   (b) How would you extend this signalling scheme to the M-ary problem of transmitting binary strings of length $r = log_2 M$ ? what role could you see the DFT playing?

10. Modify the MATLAB program Monte Carlo in Appendix 2E of Kay's detection theory to determine $\Pr\left\{(1/N)\sum_{n=0}^{N-1} x^2[n] > 5\right\}$ if $x^2[n] \sim \mathcal{N}(0, 5)$, $N = 50$, and the samples are IID. Compare your results against the theoretical probability. Hint: Use the central limit theorem.

11. A detection probability $P_D$ is to be determined via a Monte Carlo computer simulation. It is known that $P_D \geq 0.8$. If the relative absolute error is to be no more than 0.01 for 95% of the time, how many Monte Carlo trials are required?

12. We wish to detect a DC level $A$ in WGN with variance $\sigma^2$ based on the samples $x[n]$ for $n$=0,1,2....,$N$-1. The amplitude of the DC level $A$ is known but the variance $\sigma^2$ of the noise is unknown. Use a Bayesian approach with the prior PDF

$$X(\sigma^2) = \begin{cases} \frac{\lambda exp(-\lambda/\sigma^2)}{\sigma^4} & \sigma^2 > 0 \\ 0 & \sigma^2 < 0 \end{cases}$$

where $\lambda > 0$. Find the detector that maximizes $P_D$ for a fixed $P_{FA}$. Do not evaluate the threshold. Explain what happens as $\lambda \to 0$. Hint: you will need the Gamma integral

$$\int_0^\infty x^{b-1} exp(-ax) dx = a^{-b}\Gamma(b)$$

13. Find the NP test to distinguish between the hypotheses that a single sample $x[0]$ is observed from the possible PDFs

$$H_0 : p(x[0]) = \frac{1}{2} exp(-|x[0]|)$$

$$H_1 : p\left(x\left[0\right]\right) = \frac{1}{\sqrt{2\pi}} exp\left(-x^2\left[0\right]\right)$$

Show the decision regions. Hint:You will need to solve a quadratic inequality.