1. This problem illustrates what happens to an unbiased estimator when it undergoes a non-linear transformation. Consider the observations

$$x[n] = A + w[n], \quad n = 0,\, 1,\, ...,\, N\text{-}1 \tag{1}$$

where, $w[n]$ is the White Gaussian Noise (WGN). The parameter A can take any value in the interval $-\infty < A < \infty$. If we chose to estimate the unknown parameter $\theta = A^2$ by,

$$\hat{\theta} = \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2, \tag{2}$$

can we say that the estimator is unbiased? What happens as $N \to \infty$?

2. Consider the observations

$$x[n] = A + w[n], \quad n = 0,\, 1,\, ...,\, N\text{-}1 \tag{3}$$

where, $w[n]$ is the White Gaussian Noise (WGN). Lets assume that in addition to A, the value of σ^2 is also unknown. We wish to estimate the vector parameter

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} A \\ \sigma^2 \end{bmatrix}. \tag{4}$$

Is the estimator

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{A} \\ \hat{\sigma}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ \frac{1}{N-1} \sum_{n=0}^{N-1} \left(x[n] - \hat{A} \right)^2 \end{bmatrix} \tag{5}$$

unbiased?

3. We observe two samples of a DC level in *correlated* Gaussian noise

$$x[0] = A + w[0] \tag{6}$$
$$x[1] = A + w[1] \tag{7}$$

where $\mathbf{w} = [w[0]\ w[1]]^T$ is zero mean with covariance matrix

$$\mathbf{C} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{8}$$

The parameter ρ is the correlation coefficient between $w[0]$ and $w[1]$. Compute the CRLB for A and compare it to the case when $w[n]$ is WGN or $\rho = 0$. Also, explain what happens when $\rho \to \pm$. Finally, comment on the additivity property of the Fisher information for non-independent observations.

4. Consider a generalization of the line fitting problem as described in Example 3.7, termed *polynomial or curve fitting.* The data model is

$$x[n] = \sum_{k=0}^{p-1} A_k n^k + w[n] \tag{9}$$

for n = 0, 1, ..., N-1. As before, $w[n]$ is WGN with variance σ^2. It is desired to estimate $\{A_0, A_1, ..., A_{p-1}\}$. Find the Fisher information matrix for this problem.

5. Independent bivariate Gaussian samples $\{\mathbf{x}[0], \mathbf{x}[1], ..., \mathbf{x}[N-1]\}$ are observed. Each observation is a 2×1 vector which is distributed as $\mathbf{x}[n] \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ and

$$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{10}$$

Find the CRLB for the correlation coefficient ρ.

6. A line array of antennas with $d = \lambda/2$ spacing for a 1 MHz electromagnetic signal is to be designed to yield a 5^o standard deviation at $\beta = 90^o$. If the SNR η is 0 dB, comment on the feasibility of the requirement. Assume that the line array is to be mounted on the wing of an aircraft. Use $c = 3\times10^8$ m/s.

7. Consider the observation matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix} \tag{11}$$

where ϵ is small. Compute $(\mathbf{H}^T\mathbf{H})^{-1}$ and examine what happens as $\epsilon \to 0$. If $\mathbf{x} = [2\ 2\ 2]^T$, find the MVU estimator and describe what happens as $\epsilon \to 0$.

8. In the linear model it is desired to estimate the signal $\mathbf{s} = \mathbf{H}\boldsymbol{\theta}$. If an MVU estimator of $\boldsymbol{\theta}$ is found, then the signal may be estimated as $\hat{\mathbf{s}} = \mathbf{H}\hat{\boldsymbol{\theta}}$. What is the PDF of $\hat{\mathbf{s}}$?

9. In practice we sometimes encounter the "linear model" $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ but *with* $\mathbf{H}$ *composed of random variables*. Suppose we ignore this difference and use our usual estimator

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}, \tag{12}$$

where we assume that the particular realization of $\mathbf{H}$ is known to us. Show that if $\mathbf{H}$ and $\mathbf{w}$ are independent, the mean and covariance of $\hat{\boldsymbol{\theta}}$ are

$$E(\hat{\boldsymbol{\theta}}) = \boldsymbol{\theta} \tag{13}$$

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \sigma^2 E_H\left[(\mathbf{H}^T\mathbf{H})^{-1}\right] \tag{14}$$

where E_H denotes the expectation with respect to the PDF of $\mathbf{H}$. What happens if the independence assumption is not made?

10. If $x[n] = A + w[n]$ for $n = 0, 1, 2.....N-1$ are observed and

$$w = [w[0]\ w[1]\ w[2]\ ....\ w[N-1]]^T \ \sim\ N(0, C) \tag{15}$$

find the CRLB for A. Does an efficient estimator exist and if so, what is its variance?

11. For a 2 X 2 firsher information matrix $I = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ which is positive definite, show that

$$[I^{-1}(\theta)]_{11} = \frac{c}{ac-b^2} \geq \frac{1}{a} = \frac{1}{[I(\theta)]_{11}} \tag{16}$$

What does this say about estimating a parameter when a second parameter is either known or unknown? When does equality hold and why?