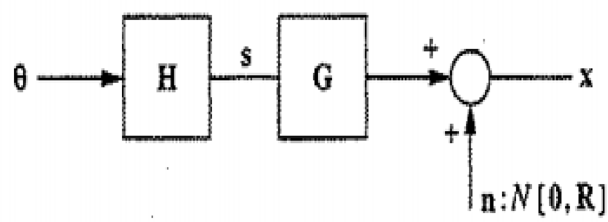
1. Consider the experimental setup shown in the figure. The problem is to observe x and estimate s. Redraw this diagram in the form of Figure 1.2 (Scharf) to show that it fits our structure of statistical reasoning. Can you describe an experiment where this diagram applies?



2. Consider the Estimator discussed in class (Section 1.4 of Scharf). Assume that θ is an unknown constant and the noises are drawn from a sequence of i.i.d. $N[0, \sigma^2]$ random variables.

   (a) Show that the estimator $\hat{\theta}_N$ is distributed as follows

   $$\hat{\theta}_N : N(\theta, \frac{\sigma^2}{N}) \tag{1}$$

   (b) Show that the estimator error is distributed as

   $$\epsilon_N : N(0, \frac{\sigma^2}{N}) \tag{2}$$

   (c) Show that the probability that $|\epsilon_N|$ exceeds $\epsilon(> 0)$ is

   $$P\left[|\epsilon_N| > \epsilon\right] \quad = \quad 2\int_{-\infty}^{-(\frac{\epsilon}{\sigma})\sqrt{N}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad = \quad 2\Phi\left(-\frac{\epsilon}{\sigma}\sqrt{N}\right) \tag{3}$$

   where

   $$\Phi\left(\eta\right) = \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{4}$$

3. Figure shows p radiating sources r_i with $i = 1, 2, ...p$ transmitting plane waves that are sensed by N sensors labeled $S_0, S_1, ...S_{N-1}$. Sensor S_0 is placed at the center of the coordinate system and the coordinate of sensor S_n is z_n.

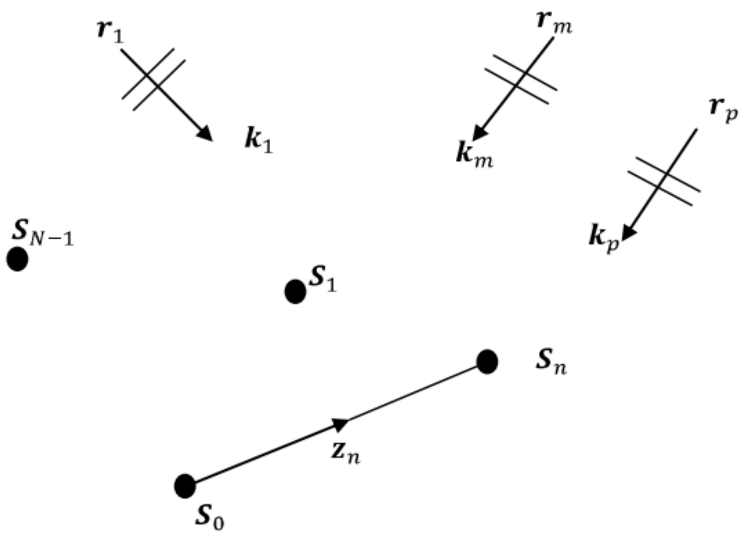
   The source transmits a propagating wave whose complex representation is

   $$r_m\left(t, z\right) = A_m e^{j\left(w_m t - k_m^T z\right)} \tag{5}$$

   The scalar frequency w_m, is the radian frequency of the source, and the vector k_m is the wave number for the source. This wave number may be written as $k_m = (2\pi/\lambda_m)d_m$, where d_m is the vector of direction cosines.

   The scalar waveform sensed by sensor S_n is the sum of all signals $r_m\left(t, z\right)$ read at $z = z_n$:

   $$x_n(t) = \sum_{m=1}^{p} r_m\left(t, z\right) \tag{6}$$



Express the received waveform vector X as

$$\begin{bmatrix} x_0(t) \\ x_1(t) \\ \dots \\ \dots \\ x_{N-1}(t) \end{bmatrix} = H \begin{pmatrix} A_1 e^{jw_1 t} \\ A_2 e^{jw_2 t} \\ .. \\ .. \\ A_p e^{jw_p t} \end{pmatrix} \tag{7}$$

where column $h_m m = 1, 2, ...p$ of matrix H characterizes the phase delays of source r_m to each of the sensors $S_n n = 0, 1, 2....N-1$

If samples are taken in time intervals of T at each sensor, then show that

$$\begin{bmatrix} x_0(0) & .. & .. & .. & x_0((M-1)T) \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ x_{N-1}(0) & .. & .. & .. & x_{N-1}((M-1)T) \end{bmatrix} = HAT \tag{8}$$

Where A is the diagonal matrix containing the source amplitudes A_m and T is a row Vandermonde matrix determined by the source frequencies w_m .

4. Let $\{x_k(\xi)\}_{k=1}^{4}$ be four IID random variables with exponential distribution (P.1, Manolakis, Chapter 3) with a = 1. Let

$$y_k(\xi) = \sum_{l=1}^{k} x_l(\xi) \quad 1 \le k \le 4 \tag{9}$$

(a) Determine and plot the pdf of $y_2(\xi)$

(b) Determine and plot the pdf of $y_3(\xi)$

(c) Determine and plot the pdf of $y_4(\xi)$

(d) Compute the pdf of $y_4(\xi)$ with that of gaussian density

5. The Cauchy distribution with mean μ is given by

$$f_x(x) = \frac{1}{\pi}\frac{1}{1+(x-\mu)^2} \qquad -\infty < x < \infty \tag{10}$$

Let $\{x_k(\zeta)\}_{i=k}^{N}$ be N IID random variables with the above distribution. Consider the mean estimator based on $\{x_k(\zeta)\}_{i=k}^{N}$

$$\hat{\mu}(\zeta) = \frac{1}{N}\sum_{k=1}^{N} x_k(\zeta) \tag{11}$$

Determine whether $\hat{\mu}(\zeta)$ is a consistent estimator of μ.

6. The mean and the covariance of a Gaussian random vector $\mathbf{x}$ are given by, respectively,

$$\boldsymbol{\mu}_{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}_{\mathbf{x}} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{12}$$

Plot the 1σ, 2σ and 3σ concentration ellipses representing the contours of the density function in the (x_1, x_2) plane. *Hints:* The radius of an ellipse with major axis a (along x_1) and minor axis $b < a$ (along x_2) is given by

$$r^2 = \frac{a^2b^2}{a^2sin^2\theta + b^2cos^2\theta} \tag{13}$$

where $0 \le \theta \le 2\pi$. Compute the 1σ ellipse specified by $a = \sqrt{\lambda_1}$ and $b = \sqrt{\lambda_2}$ and then rotate and translate each point $\mathbf{x}^{(i)} = \begin{bmatrix} x_1^{(i)} & x_2^{(i)} \end{bmatrix}^T$ using the transformation $\mathbf{w}^{(i)} = \mathbf{Q}_{\mathbf{x}}\mathbf{x}^{(i)} + \boldsymbol{\mu}_{\mathbf{x}}$.

7. A causal LTI system, which is described by the difference equation

$$y(n) = \frac{1}{2}y(n-1) + x(n) + \frac{1}{3}x(n-1) \tag{14}$$

is driven by a zero-mean WSS process with auto-correlation $r_x(l) = 0.5^{|l|}$.

(a) Determine the PSD and auto-correlation of the output sequence $y(n)$.

(b) Determine the cross-calibration $r_{xy}(l)$ and cross-PSD $R_{xy}(e^{j\omega})$ between the input and the output signals.

(Illustrate Results in MATLAB)

8. Show that

$$E[(aX+b)^n] = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i}b^iE(X^{n-i}) \tag{15}$$

9. Consider a normal random vector $\mathbf{x}(\xi)$ with components that are mutually uncorrelated, that is, $\rho_{ij} = 0$. Show that (a) the covariance matrix Γ_x is diagonal and (b) the components of $\mathbf{x}(\xi)$ are mutually independent.

10. Determine whether the following matrices are valid correlations matrces:

$$(a) \quad \mathbf{R}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad (b) \quad \mathbf{R}_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

$$(c) \quad \mathbf{R}_3 = \begin{bmatrix} 1 & 1\text{-}j \\ 1+j & 1 \end{bmatrix} \qquad (d) \quad \mathbf{R}_4 = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

11. A WSS process with PSD $R_x(e^{j\omega}) = 1/(1.64 + 1.6\ cos\omega)$ is applied to a causal system described by the following difference equation

$$y(n) = 0.6y(n-1) + x(n) + 1.25x(n-1)$$

Compute (a) the PSD of the output and (b) the cross-PSD $R_{xy}(e^{j\omega})$ between input and output.

12. For each of the following, determine whether the random process is (1) WSS or (2) m.s.ergodic in the mean.

    (a) $X(t) = A$, where A is a random variable uniformly distributed between 0 and 1.
    (b) $X_n = A\ \cos\omega_0 n$, where A is a Gaussian random variable with mean 0 and variance 1.
    (c) A Bernoulli process with $\Pr[X_n = 1]{=}p$ and $\Pr[X_n = -1]{=}1-p$