Assignment No. 1

1. Consider the correlation detector discussed in class. Assume that the symbol $\{s_t\}_1^N$ is known, with energy E*s*, and assume the noises $\{n_t\}_1^N$ are drawn from a sequence of independent, identically distributed (i.i.d.) normal (or Gaussian) random variables. That is,

$$n_t = N[0, \sigma^2]$$

$$E(n_t) = 0 \; ; \quad var(n_t) = \sigma^2.$$

   a. Show that the correlation statistic is distributed as follows under $H_0 : \theta = -\mu$ and $H_1 : \theta = \mu$:

$$c_N : \; N[-\mu Es, \sigma^2 Es] \text{ under } H_0$$

$$c_N : \; N[\mu Es, \sigma^2 Es] \text{ under } H_1$$

   Plot these normal densities. (Implement on MATLAB)

   b. Define the output SNR of the correlation statistic to be the mean squared, divided by the variance. Show

$$\text{SNR} = \frac{\mu^2}{\sigma^2} \text{E}s.$$

   c. If the alternative H_1 is selected when $c_N > 0$, show that the probability of falsely choosing H_1 is

$$P[H_1|H_2] = \int_{(\mu/\sigma)\sqrt{E_s}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \int_{-\infty}^{(\mu/\sigma)\sqrt{E_s}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 1 - \Phi\left(\frac{\mu}{\sigma}\sqrt{E_s}\right)$$

$$\Phi(\eta) = \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad : \quad \text{normal integral.}$$

2. Consider the Estimator discussed in class (Section 1.4 of Scharf) Assume that θ is an unknown constant and the noises are drawn from a sequence of i.i.d. $N[0, \sigma^2]$ random variables.

   a. Show that the estimator $\hat{\theta}_N$ is distributed as follows

$$\hat{\theta}_N : N[\theta, \frac{\sigma^2}{N}].$$

b. Show that the estimator error is distributed as

$$\epsilon_N : N[0, \frac{\sigma^2}{N}].$$

c. Show that the probability that $|\epsilon_N|$ exceeds $\epsilon > 0$ is

$$P[|\epsilon_N| > \epsilon] = 2 \int_{-\infty}^{-(\epsilon/\sigma)\sqrt{N}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= 2\,\Phi\left(-\frac{\epsilon}{\sigma}\sqrt{N}\right)$$

$$\Phi(\eta) = \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

3. Figure shows p radiating sources r_i transmitting plane waves that are sensed by N sensors labeled S_0, S_1,…S_{N-1}. Sensor S_0 is placed at the center of the coordinate system, and the coordinate of sensor S_n is $\mathbf{z}_n$. The source r_m transmits a propagating wave whose complex representation is

$$r_m(t, \mathbf{z}) = A_m e^{j(\omega_m t - \boldsymbol{k}_m^T \boldsymbol{z})} = A_m e^{j\omega_m t} e^{-j\boldsymbol{k}_m^T \boldsymbol{z}}$$

The scalar frequency ω_m is the radian frequency of the source, and the vector $\boldsymbol{k}_m$ is the wave number for the source. This wave number may be written as $\boldsymbol{k}_m = (2\pi/\lambda_m)\boldsymbol{d}_m$, where $\boldsymbol{d}_m$ is the vector of direction cosines. The field propagating from radiator r_m may now be written as

$$r_m(t, \mathbf{z}) = A_m e^{j\omega_m t} e^{-j(2\pi/\lambda_m)\boldsymbol{d}_m^T \boldsymbol{z}}.$$

The scalar waveform sensed by sensor S_n is the sum of all signals $r_m(t, \mathbf{z})$, read at $z = \mathbf{z}_n$:

$$x_n(t) = \sum_{m=1}^{p} r_m(t, \mathbf{z}).$$

Express the received waveform vector x as

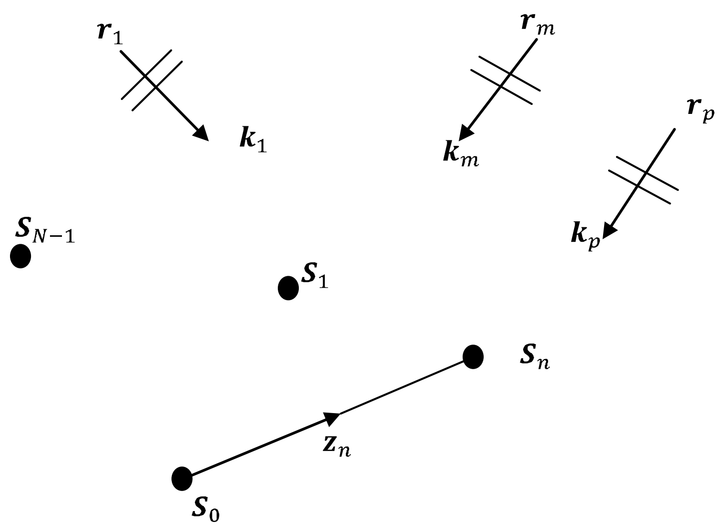
$$\begin{bmatrix} x_0(t) \\ \vdots \\ x_{N-1}(t) \end{bmatrix} = H \begin{pmatrix} A_1 e^{j\omega_1 t} \\ \vdots \\ A_p e^{j\omega_p t} \end{pmatrix}$$

where column $\boldsymbol{h}_m$, $m = 1, 2, \ldots, p$ of matrix $\mathbf{H}$ characterizes the phase delays of source r_m to each of the sensors S_n, $n = 0, 1, \ldots, N-1$.

If samples are taken in time intervals of T at each sensor, then show that

$$\begin{bmatrix} x_0(0) & \cdots & x_0[(\text{M-1})\text{T}] \\ \vdots & \ddots & \vdots \\ x_{N-1}(0) & \cdots & x_{N-1}[(\text{M-1})\text{T}] \end{bmatrix} = \mathbf{HAT}$$

Where **A** is the diagonal matrix containing the source amplitudes A_m and **T** is a row Vandermonde matrix determined by the source frequencies ω_m.



4. It is desired to estimate the value of a DC level A in WGN or

$$\text{x}[n] = A + \text{w}[n] \qquad \text{n} = 0, 1, \ldots, \text{N-1}$$

where w$[n]$ is zero mean and uncorrelated, and each sample has variance $\sigma^2 = 1$

Consider the estimator

$$\hat{A} = \begin{cases} x[0] & \frac{A^2}{\sigma^2} = A^2 > 1000 \\ \frac{1}{N}\sum_{n=0}^{N-1} x[n] & \frac{A^2}{\sigma^2} = A^2 \le 1000 \end{cases}$$

The rationale for this estimator is that for a high enough SNR or A^2/σ^2, we do not need to reduce the effect of noise by averaging and hence avoid the added computation. Comment on this approach.